GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 - OCTOBER 2015

ROUND 1 – Arithmetic - Open

1. ______________________ feet

2. (__________ , __________)

3. ______________________

1. Using what he assumed was a yardstick, Al measured the length of a wall and thought it was 216 feet in length. Later, it was found that the measuring stick he used was two inches shorter than a yard. What is the actual length of the wall in feet?

   Note: 1 yard = 3 feet

2. Determine the ordered pair (x, y), if $0.13\overline{13}_{(\text{base } 5)} = 0.xy\overline{xy}_{(\text{base } 8)}$.

3. How many natural numbers between 244 and 870 are divisible by 7 or 11?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2015

ROUND 2 – Algebra 1 – Word Problems

1. ______________________

2. X: __________ Y: __________

3. (__________ , __________)

1. Compute the <u>largest</u> of three consecutive positive even integers denoted by $(x-2, x, x+2)$ such that their sum is $\frac{1}{132}$ of their product.

2. Mr. X can do a job in 5 hours and Mr. Y can do the same job in 8 hours. Mr. X works alone for a period of time and stops. At this time, Mr. Y starts and finishes the job. The total time worked by both men was $6\frac{2}{3}$ hours. Compute the time (in hours) each of the men worked.

3. On the Farrahite scale, water freezes at 40°F and boils at 280°F. On the Centistade scale, water freezes at 25°C and boils at 125°C. Assuming a linear relationship between the two temperature scales, $F = mC + b$. Compute the ordered pair (m, b).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2015

ROUND 3 – Algebra 1 – Exponents and Radicals

1. ______________________

2. ______________________

3. ______________________

1. Simplify completely: $\sqrt[4]{4}\cdot\sqrt[6]{8}\cdot\left(\frac{8}{27}\right)^{-\frac{1}{3}}$

2. Simplify completely: $\dfrac{\left(\sqrt{3}+\sqrt{2}-\sqrt{6}\right)\left(\sqrt{3}+\sqrt{2}+\sqrt{6}\right)}{1-2\sqrt{6}}$

3. For what value(s) of x are both of the following statements true?

$$5\cdot 2^{x-2}=\frac{4}{5}y \text{ and } 2\cdot 5^{x+1}=625y$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2015

ROUND 4 – Algebra 2 – Factoring

1. ________________________

2. ________________________

3. ________________________

1. Factor completely: $(x-1)^4 - 2(x-1)^2 - 8$

2. Factor completely: $9y^3 - 3xy^2 + 3x^3 - 9x^2y$

3. Compute the ordered pair (p,q), if $x^2 + 5x + 4$ is a factor of $x^3 + 3x^2 + 3px + q$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2015

ROUND 5 – Trig: Angular and Linear Velocity, Right Triangles

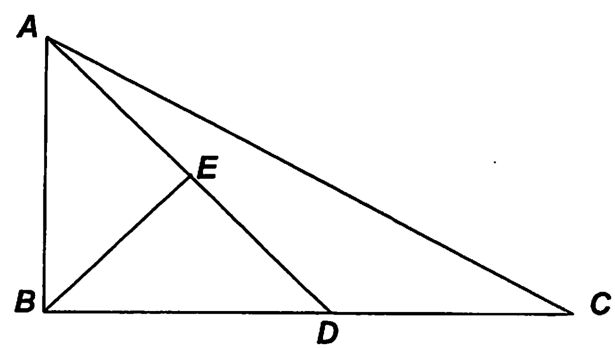
1. ______________________

2. ______________________

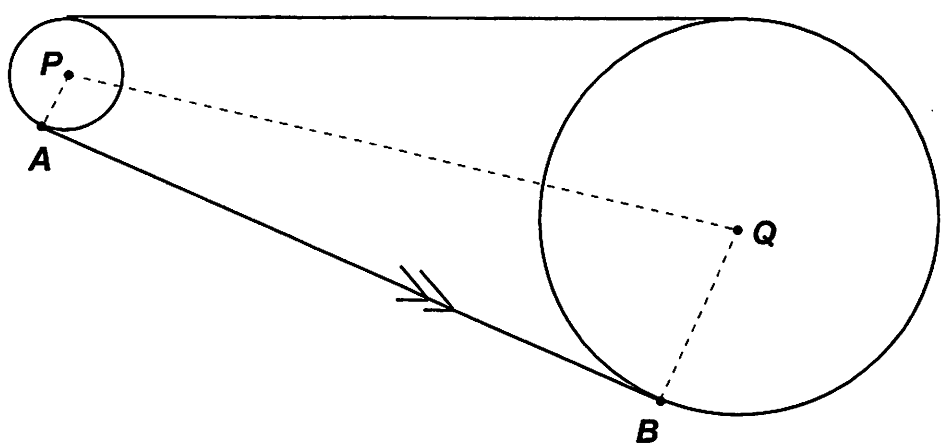
3. ______________________

1. Compute the number of <u>radians</u> through which the hour hand of a regular 12-hour clock moves in one hour and 48 minutes.

2. Given: $\overline{AB} \perp \overline{BC}$, $\overline{BE} \perp \overline{AD}$, $m\angle BDE = 45°$
   $m\angle BCA = 30°$
   If the area of ΔBDE is 6,
   compute the perimeter of ΔABC.



3. A belt is moving counterclockwise (without slipping) around two circular pulleys: P with radius 2 feet, and Q with radius 30 feet. If pulley P turns at k revolutions per minute (RPMs) and $PQ = 100$ feet, a mark on the belt at point A moves past point B in 1 second. A and B are points of tangency of the belt with pulleys P and Q respectively. Using the approximation $\pi = \dfrac{22}{7}$, compute k to the nearest 5 RPMs.



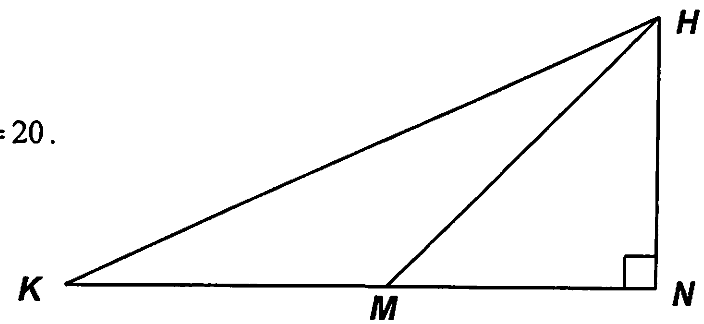
GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2015

TEAM ROUND

3 pts.	1.	______________
3 pts.	2.	______________
4 pts.	3.	______________

1. Given right triangle *KNH* with $m\angle HMN = 2m\angle K$, $\sin K = 0.6$, $KH = 20$. Compute *KM*.



2. A boy has *N* marbles. When he puts them in piles of three, four, six, seven, or eight marbles, there is always one marble left over. When he puts them in piles of eleven marbles, there are none left over. Compute the <u>smallest</u> possible value of *N*.

3. Determine the <u>number</u> of ordered quadruples of positive integers (A,B,C,D) whose sum is less than 20 and which satisfy these inequalities $\begin{cases} A > B \\ C > A + B \\ D > A + B + C \end{cases}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2015

Answer Sheet

Round 1

1. 204
2. (2, 5)
3. 139

Round 2

1. 22
2. $\left(\frac{20}{9},\frac{40}{9}\right)$
3. $\left(\frac{12}{5},-20\right)$ or equivalent

Round 3

1. 3
2. –1
3. 5

Round 4

1. $(x-3)(x+1)(x^2-2x+3)$
2. $3(x-3y)(x+y)(x-y)$ or $3(3y-x)(y+x)(y-x)$
3. $(-2,-8)$

Round 5

1. $\frac{3\pi}{10}$
2. $6\left(\sqrt{2}+\sqrt{6}\right)$ or equivalent
3. 460

Team Round

1. 12.5 (3 pts)
2. 1177 (3 pts)
3. 10 (4 pts)

Detailed Solutions for GBML Meet 1 – OCTOBER 2015

ROUND 1

1. To measure 216 feet, the yardstick was applied $\frac{216}{3}=72$ and, since the yardstick was really only 34 inches long, there was a loss of 2 inches each time. 144 inches is equivalent to 12 feet and the actual length of the wall is **204** feet.

2. Let $N_{(10)} = 0.13\overline{13}_{(5)}$. To move the "decimal" point 2 places, we must multiply the right side of the equation by $100_{(5)}$ and the left side by the base 10 equivalent, namely 25.

$$\begin{cases} 25N_{(10)} = 13.\overline{13}_{(5)} \\ N_{(10)} = 0.\overline{13}_{(5)} \end{cases} \Rightarrow 24N_{(10)} = 13_{(5)} = 1(5)+3 = 8 \Rightarrow N_{(10)} = \frac{1}{3}$$

Similarly, $M_{(10)} = 0.xy\overline{xy}_{(\text{base } 8)} \Rightarrow M_{(10)} = \frac{8x+y}{63}$.

$\frac{1}{3} = \frac{21}{63} \Rightarrow 8x+y=21$. As digits in base 8, the only solution is $(x,y)=\underline{\mathbf{(2,5)}}$.

3. Since $245=7(35)$ and $868=7(124)$, there are $124-35+1=90$ multiples of 7 between 244 and 870. Since $253=11(23)$ and $869=11(79)$, there are $79-23+1=57$ multiples of 11. However, there is an overlap, since multiples of 77 are included in both of these counts. Since $231=77(3)$, $308=77(4)$ and $847=77(11)$, there are $11-4+1=8$ multiples of 77. Therefore, there are $90+57-8=$ **139** multiples of 7 or 11 in the specified interval.

Detailed Solutions for GBML Meet 1 – OCTOBER 2015

ROUND 2

1. The sum is $3x$ and the product is $x\left(x^2-4\right)$.

   Thus, $3x=\frac{1}{132}x\left(x^2-4\right)\Rightarrow 396=x^2-4\Rightarrow x^2=400\Rightarrow x=20\Rightarrow$ largest: $x+2=\underline{\underline{22}}$.

2. Suppose Mr. X works for A hours before quitting and Mr. Y works for $B=\left(\frac{20}{3}-A\right)$ hours to complete the job. Then:

   The fraction of the job completed by Mr. X is $A\left(\frac{1}{5}\right)$ and Mr. Y completes $\left(\frac{20}{3}-A\right)\left(\frac{1}{8}\right)$.

   $A\left(\frac{1}{5}\right)+\left(\frac{20}{3}-A\right)\left(\frac{1}{8}\right)=1\Rightarrow\frac{A}{5}+\frac{20-3A}{24}=1$

   $\Rightarrow 24A+100-15A=120\Rightarrow 9A=20\Rightarrow$ X: $\underline{\mathbf{\frac{20}{9}}}$ hours Y: $\underline{\mathbf{\frac{40}{9}}}$ hours

3. $F=mC+B\Rightarrow\begin{cases}40=25m+b\\280=125m+b\end{cases}$

   Subtracting, $240=100m\Rightarrow m=\frac{12}{5}$

   Substituting, $b=40-25\left(\frac{12}{5}\right)=40-60=-20$

   Thus, $(m,b)=\underline{\mathbf{\left(\frac{12}{5},-20\right)}}$ or equivalent.

Detailed Solutions for GBML Meet 1 – OCTOBER 2015

ROUND 3

1. $\sqrt[4]{4} = 4^{\frac{1}{4}} = \left(2^2\right)^{\frac{1}{4}} = 2^{\frac{1}{2}} = \sqrt{2}$

   Similarly, $\sqrt[6]{8} = \left(2^3\right)^{\frac{1}{6}} = \sqrt{2}$

   $\left(\frac{8}{27}\right)^{-\frac{1}{3}} = \left(\frac{27}{8}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{27}{8}} = \sqrt[3]{\frac{3^3}{2^3}} = \frac{3}{2}$

   Thus, $\sqrt[4]{4}\cdot\sqrt[6]{8}\cdot\left(\frac{8}{27}\right)^{-\frac{1}{3}} = \sqrt{2}\cdot\sqrt{2}\cdot\frac{3}{2} = \underline{3}$.

2. Recognizing the numerator as the product of a sum and a difference, we apply the identity $(X-Y)(X+Y) = X^2 - Y^2$.

   $$\frac{\left(\sqrt{3}+\sqrt{2}-\sqrt{6}\right)\left(\sqrt{3}+\sqrt{2}+\sqrt{6}\right)}{1-2\sqrt{6}} = \frac{\left(\sqrt{3}+\sqrt{2}\right)^2-\left(\sqrt{6}\right)^2}{1-2\sqrt{6}} =$$

   $$\frac{3+2+2\sqrt{6}-6}{1-2\sqrt{6}} = \frac{-1+2\sqrt{6}}{1-2\sqrt{6}} = \frac{-1\left(\cancel{1-2\sqrt{6}}\right)}{\cancel{1-2\sqrt{6}}} = \underline{\mathbf{-1}}$$

3. $5\cdot 2^{x-2} = \frac{4}{5}y \Leftrightarrow y = 25\left(\frac{2^{x-2}}{4}\right) = 25\left(2^{x-4}\right)$

   $2\cdot 5^{x+1} = 625y \Leftrightarrow y = \frac{2\cdot 5^{x+1}}{625} = \frac{2\cdot 5^{x+1}}{5^4} = 2\cdot 5^{x-3}$

   Equating, $25\left(2^{x-4}\right) = 2\cdot 5^{x-3} \Leftrightarrow \frac{2^{x-4}}{2} = \frac{5^{x-3}}{25} \Leftrightarrow 2^{x-5} = 5^{x-5}$

   But this is only possible if the exponents are zero!
   Thus, $x = \underline{5}$.

Detailed Solutions for GBML Meet 1 – OCTOBER 2015

ROUND 4

1. Let $Y=(x-1)^2$. Then: $(x-1)^4-2(x-1)^2-8 \Leftrightarrow Y^2-2Y-8 \Leftrightarrow (Y-4)(Y+2)$

   Substituting, we have $\left((x-1)^2-4\right)\left((x-1)^2+2\right)=\left(x^2-2x-3\right)\left(x^2-2x+3\right)$

   Only the first trinomial factors over the integers.

   $\Rightarrow \underline{\mathbf{(x-3)(x+1)(x^2-2x+3)}}$

   The order of the factors is irrelevant.

2. Grouping terms in pairs, $9y^3-3xy^2+3x^3-9x^2y=\left(9y^3-3xy^2\right)+\left(3x^3-9x^2y\right)$

   Extracting common factors from each binomial, $3y^2(3y-x)+3x^2(x-3y)$

   $\Rightarrow 3y^2(3y-x)-3x^2(3y-x)$

   $\Rightarrow (3y-x)\left(3y^2-3x^2\right)=3(3y-x)\left(y^2-x^2\right)$

   $\Rightarrow \underline{\mathbf{3(y-x)(y+x)(3y-x)}}$ or equivalent e.g. $\underline{\mathbf{3(x+y)(x-y)(x-3y)}}$

3. Since $x^2+5x+4=(x+4)(x+1)$, we must find p and q so that

   $x^3+3x^2+3px+q=(x+4)(x+1)(x-c)$

   Rather than multiplying out the right hand side and equating the coefficients of like terms, we choose to substitute for x.

   $x=-1 \Rightarrow -1+3-3p+q=0$ and $x=-4 \Rightarrow -64+48-12p+q=0$

   $$\begin{cases}-3p+q=-2\\-12p+q=16\end{cases} \Rightarrow 9p=-18 \Rightarrow p=-2$$

   Thus, $(p,q)=\underline{\mathbf{(-2,-8)}}$

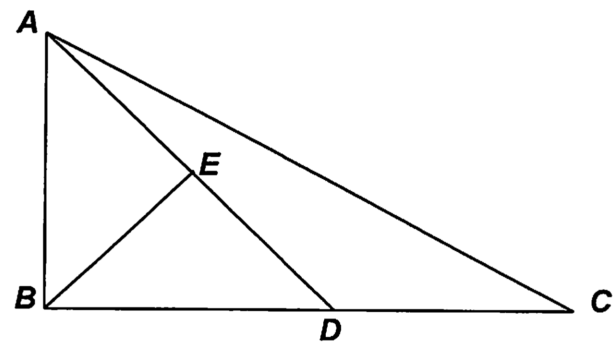
   Check: Clearly, $-4c=-8 \Rightarrow c=2$.

   If is left to you to verify that $x^3+3x^2-6x-8=(x+4)(x+1)(x-2)$.

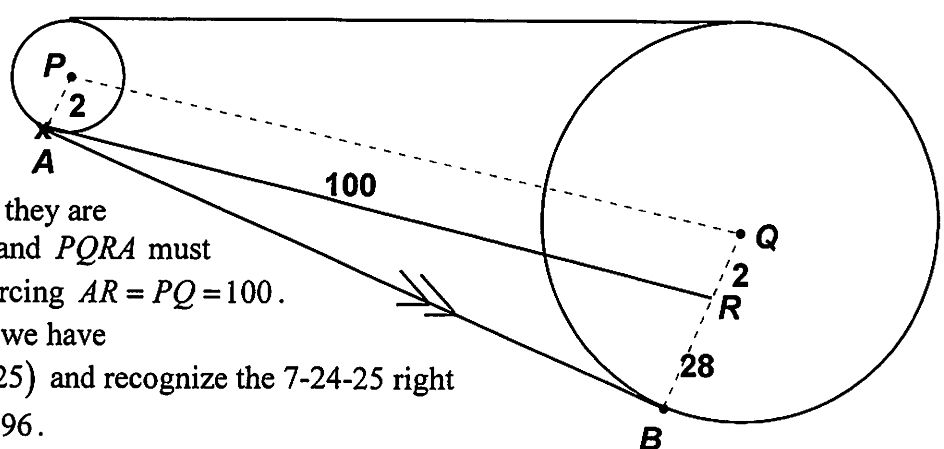
Detailed Solutions for GBML Meet 1 – OCTOBER 2015

ROUND 5

1. In one hour, the hour hand turns $\frac{1}{12}$ of a revolution or $\frac{1}{12}\cdot 360° = 30° = \frac{\pi}{6}$ radians
   One hour and 48 minutes is $1\frac{48}{60} = \frac{9}{5}$ hour and the hand turns through $\frac{9}{5}\cdot\frac{\pi}{6} = \underline{\mathbf{\frac{3\pi}{10}}}$.



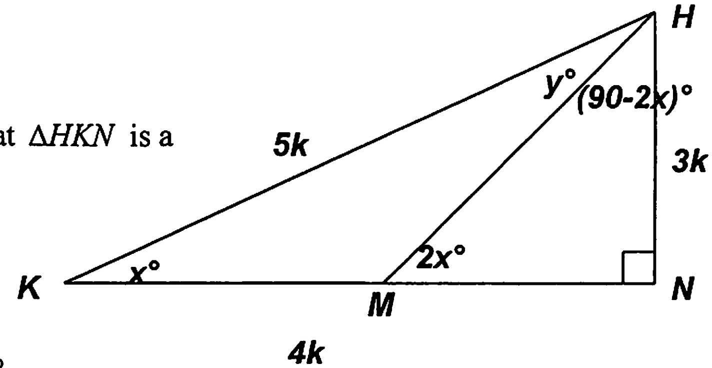
2. If $m\angle BDE = 45°$ and $\overline{BE} \perp \overline{AD}$, then
   ΔBED is an isosceles right triangle.
   Similarly, ΔABD is isosceles with $AB = BD$
   Let $BE = DE = x$. Then: $BD = x\sqrt{2}$
   $\text{Area}(\Delta BED) = 6 \Rightarrow \frac{1}{2}x^2 = 6 \Rightarrow x = 2\sqrt{3}$
   $\Rightarrow BD = AD = 2\sqrt{6}$
   $m\angle C = 30° \Rightarrow \Delta ABC$ is a 30-60-90 triangle and $AC = 4\sqrt{6}$, $BC = 2\sqrt{6}\cdot\sqrt{3} = 6\sqrt{2}$
   Thus, the perimeter of ΔABC is $\underline{\mathbf{6\sqrt{6}+6\sqrt{2}}}$ or $\underline{\mathbf{6\left(\sqrt{6}+\sqrt{2}\right)}}$.



3. Draw $\overline{AR}$ parallel to $\overline{PQ}$. Since $\overline{PA}$ and $\overline{QB}$ are both perpendicular to $\overline{AB}$, they are parallel to each other and $PQRA$ must be a parallelogram, forcing $AR = PQ = 100$.
   In right triangle *BAR*, we have $(?, 28, 100) = 4(\_\_, 7, 25)$ and recognize the 7-24-25 right triangle, forcing $AB = 96$.
   One revolution around pulley *P* is 4π feet.
   k RPM $= 4\pi k \frac{\text{ft}}{\text{min}} = 4\pi k \frac{\text{ft}}{60 \text{ sec}} = \frac{\pi k}{15}\frac{\text{ft}}{\text{sec}}$. Thus, in one second, the mark "x" at point *A* must move past point *B* and we have $\frac{\pi k}{15} > 96 \Rightarrow k > \frac{1440}{\pi} \approx \frac{1440(7)}{22} \approx \frac{5040}{11} \approx 458^{+} \Rightarrow k = \underline{\mathbf{460}}$ RPM

Detailed Solutions for GBML Meet 1 – OCTOBER 2015

TEAM ROUND



1. Since $\sin K = \sin x° = 0.6 = \frac{3}{5}$, we note that ΔHKN is a 3-4-5 right triangle.
   $HK = 20 \Rightarrow k = 4 \Rightarrow (KN, HN) = (16, 12)$
   and $\cos K = \frac{4}{5}$.
   Since $\sin\angle HMN = \sin 2x° = 2\sin x° \cos x°$,
   $\frac{HN}{HM} = \frac{12}{HM} = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25} \Rightarrow HM = 12.5$.
   But, in right triangle *KHN*, $\angle K$ and $\angle KHN$ are complementary, and we have
   $x° + y° + (90 - 2x)° = 90° \Rightarrow y = x$ and ΔKHM is isosceles. Therefore, $KM =$ **<u>12.5</u>**.

   Note: By the exterior angle theorem, $\angle HMN = 2x° = m\angle HKM + m\angle HMK = x° + y° \Rightarrow x = y$, ΔKMH must be isosceles, and the same result follows.

2. Since the least common multiple of 3,4,6,7, and 8 is $2^3 \cdot 3 \cdot 7 = 168$, $N = 168k + 1$. Examining numbers of this form, we have 169, 337, 505, 673, 841, 1009, 1177, 1345.
   Applying the divisibility by 11 rule, we are looking for a multiple of 11, e.g. $0, \pm 11, \pm 22, \ldots$.
   $169 \Rightarrow (1+9) - 6 = 4$ $\quad 337 \Rightarrow (3+7) - 3 = 7$ $\quad 505 \Rightarrow (5+5) - 0 = 10$
   $673 \Rightarrow (6+3) - 7 = 2$ $\quad 841 \Rightarrow (8+1) - 4 = 5$ $\quad 1009 \Rightarrow (1+0) - (0+9) = -8$
   <u>1177</u> $\Rightarrow (1+7) - (1+7) = 0$ Bingo!

3. The smallest possible sum of $A + B + C$ is 7 for $(A, B, C) = (2, 1, 4)$ since $\begin{cases} A > B \\ C > A + B \end{cases}$
   and *A*, *B*, and *C* must be positive integers. Since $A + B + C + D < 20$ and $D > A + B + C$, the largest and smallest values of *D* are 12 and 8 respectively.
   Thus, there are **<u>10</u>** possible ordered quadruples (A, B, C, D), namely,
   $(2,1,4,12)$, $(2,1,4,11)$, $(2,1,5,11)$, $(2,1,6,10)$, $(2,1,5,10)$,
   $(2,1,4,10)$, $(3,1,5,10)$, $(2,1,5,9)$, $(2,1,4,9)$, $(2,1,4,8)$