GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 - OCTOBER 2017

ROUND 1 - Arithmetic: Open

1. ______________________

2. ______________________

3. ______________________

1. Compute <u>all</u> ordered pairs (A, B), where $A+B<35$, for which $86_{(\text{base } A)} = 68_{(\text{base } B)}$.

2. Given the following array of numbers: 17, 30, 43, 56, 69, … .
Compute the <u>smallest</u> natural number in this sequence whose digits are all the same.

3. The base 10 number 30!, when expressed in base 18, ends with exactly k zeros.
Compute the value of k.

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 2 - Algebra 1: Word Problems

1. ________________________ hours

2. ____________________________

3. ________________________ hours

1. Joe can complete a job by himself in 8 hours. Dan takes 12 hours working alone to complete the same job. On Monday morning, they start at 8AM, working together. At 11:30 AM Harry joins them, and together the three men complete the job at 12:30 PM. How long (in hours) would it have taken Harry to do the job alone?

2. Presently, the sum of the ages of a father (f) and his son (s) is 33 years. In x years, the father's age will be exactly $2\frac{1}{2}$ times the son's age. If the father was in his twenties when his son was born, compute <u>all</u> possible ordered triples (x, f, s).

3. A plane travels 3900 miles in T hours at R mph. If the plane increases its speed by 100 mph, the trip takes 48 <u>minutes</u> less time. Disregarding the effect of the prevailing winds, find the travel time (in hours) for a roundtrip of 7800 miles at the plane's faster speed.

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 3 - Algebra 1: Exponents and Radicals

1. ______________________

2. ______________________

3. ______________________

1. Simplify completely. $\sqrt{\dfrac{\sqrt[12]{64}\cdot\sqrt[8]{16}}{2^{-3}-3^{-2}}}$

2. Simplify completely. $\sqrt{2\sqrt{99}+\dfrac{2}{10+3\sqrt{11}}}$

3. Compute $\left(\sqrt[4]{4}+3\sqrt[6]{27}+\sqrt[6]{125}\right)\left(3\sqrt{3}-\sqrt{5}+\sqrt{2}\right)$.

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 4 - Algebra 2: Factoring

1. ______________________________

2. ______________________________

3. ______________________________

Factor each of the following expressions completely over the integers.

1. $12x^2 - 42 - 165x$

2. $25x^2 + 105y - 49y^2 - 75x$

3. $4a^2 + a^6 - a^4 - 4$

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 5 - Trig: Angular and Linear Velocity, Right Triangles

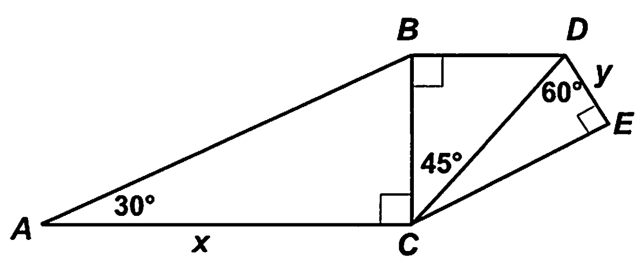
1. ______________________

2. ______________________

3. (__________ , __________)

1. Compute <u>all</u> values of x over $0° \le x < 360°$ for which $\tan(-855°)\cos(-480°)+\sin(-660°)\cot(-930°)=\sec(x)$.

2. Compute y in terms of x.



3. The wheels of an automobile travelling 45 miles per hour make 10 revolutions per second. As a simplified fraction, the diameter of each wheel is $\dfrac{a}{b\pi}$ inches. Compute the ordered pair (a,b).

GREATER BOSTON MATHEMATICS LEAGUE

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TEAM ROUND

3 pts. 1. ____________ : ____________

3 pts. 2. (____________ , ____________)

4 pts. 3. ____________________________

1. The current in a river is flowing at a rate of 1.5 miles per hour. A ferry boat travels at a constant rate in still water. The ferry's destination is 6 miles downstream. The travel time for a round trip is one hour and 57 minutes. Compute the ratio of the downstream rate to the upstream rate.

2. For the integer constants a and b, $\frac{x-a}{b}-\frac{b}{x-a}=\frac{a}{x-b}-\frac{x-b}{a}$. If $a \in \{1,2\}$ and $b \in \{3,4\}$, there are exactly N distinct x-values that satisfy the equation, k of which are integers. Compute the ordered pair (N,k).

3. The numbers 3 and 6 have a rather unusual (but not unique) property, namely, their sum is a factor of their product; specifically, $\frac{18}{9}=2$. How many ordered pairs of positive integers (x,y) are there, where $x<y$ and the quotient $\frac{xy}{x+y}$ has an integer value between 3 and 9 inclusive?

GREATER BOSTON MATHEMATICS LEAGUE

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Answer Sheet

Round 1

1. (10,13), (13,17)
2. 888
3. 7

Round 2

1. 16
2. (8, 27, 6), (15,30, 3)
3. 10.4

Round 3

1. 12
2. $2\sqrt{5}$
3. $6\left(4+\sqrt{6}\right)$ or $24+6\sqrt{6}$

Round 4

1. $3(4x+1)(x-14)$
2. $(5x-7y)(5x+7y-15)$
3. $\left(a^2+2a+2\right)\left(a^2-2a+2\right)(a+1)(a-1)$

Round 5

1. 120, 240
2. $y=\frac{\sqrt{6}}{6}x$
3. (396, 5)

Team Round

1. 8 : 5 (3 pts)
2. (8, 4) (3 pts)
3. 14 (4 pts)

Detailed Solutions for GBML Meet 1 - OCTOBER 2017

ROUND 1

1. $86_{(\text{base } A)} = 68_{(\text{base } B)} \Leftrightarrow 8A+6=6B+8 \Leftrightarrow 4A+3=3B+4 \Leftrightarrow B=\dfrac{4A-1}{3}=A+\dfrac{A-1}{3}$.

   Thus, $A-1$ must be a multiple of 3.
   $A=4 \Rightarrow B=5$.
   Since 6 and 8 are not legal digits in either base 4 or 5, these bases are rejected.
   $A=7$ is likewise rejected (even though $B=9$ would have been acceptable).
   $A=10 \Rightarrow \underline{\mathbf{(10,13)}}$.
   $A=13 \Rightarrow \underline{\mathbf{(13,17)}}$.
   $A=16 \Rightarrow$ ~~(16,21)~~, rejected since $A+B \not< 35$.

2. The sequence 17, 30, 43, 56, 69, ... is arithmetic with a common difference of 13
   The next two terms are 82 and 95; so, the natural number we seek must have at least 3 digits.
   The general term for this sequence is $13n+4$ for $n=1, 2, 3, \ldots$.
   $13n+4=100d+10d+d=111d$
   $\Rightarrow n=\dfrac{111d-4}{13}=8d+\dfrac{7d-4}{13}$
   We require $7d-4$ be a multiple of 13 for some $d=1, 2, 3, \ldots$
   The sequence generated is
   $(d,n)=(1,3),(2,10),(3,17),(4,24),(5,31),(6,38),(7,45),(8,52),\ldots$
   Thus, for $d=8$, we have the smallest positive integer in the sequence with identical digits is **<u>888</u>**.
   It is, in fact, the $8\cdot 8+\dfrac{52}{13}=68^{\text{th}}$ number in the sequence, since $13(68)+4=884+4=888$.

3. In base 10, the number of terminal zeros depends on the number of factors of 10.
   Since $10=2\cdot 5$ and there are always more factors of 2 than 5 in a factorial, the number of factors of 10 will equal the number of factors of 5.
   In base 18, the number of terminal zeros depends on the number of factors of 18.
   Since $18=2^1\cdot 3^2$, we must count the factors of 2 and 3.
   2: 2, 6, 10, 14, 18, 22, 26, 30 $\Rightarrow$ 8 factors (1 each)
   2^2: 4, 12, 20, 28 $\Rightarrow$ 8 (2 each)
   2^3: 8, 24 $\Rightarrow$ 6 (3 each)
   2^4: 16 only $\Rightarrow$ 4 Total: 26 factors of 2
   3, 6, 9, 12, 15, 18, 21, 24, 27, 30 $\Rightarrow$1+1+2+1+1+2+1+1+3+1=14 factor of 3
   14 factors of 3 and 7 factors of 2 yields 7 factors of 18 $\Rightarrow$ <u>7</u> terminal zeros in base 18.

Detailed Solutions for GBML Meet 1 - OCTOBER 2017

ROUND 2

1. We are given that Joe and Dan can complete the job in 8 and 12 hours, each working alone. Let us assume that Harry would take x hours, working alone. Thus, their respective rates are 1/8, 1/12 and $1/x$. Jan and Dan each work for 4.5 hours and Harry works for only 1 hour. Thus, adding the fractions of the job completed by each worker, the whole job gets done.
$$\frac{1}{8}\cdot 4.5+\frac{1}{12}\cdot 4.5+\frac{1}{x}\cdot 1=1$$
Multiplying by 48, we have $27+18+\frac{48}{x}=48 \Rightarrow \frac{48}{x}=3 \Rightarrow x=\underline{\mathbf{16}}$.

2. $f+s=33,\ f+x=2.5(s+x) \Leftrightarrow f+x=2.5(33-f+x)$
Multiplying out and collecting like terms, $2f+2x=165-5f+5x \Leftrightarrow 7f-3x=165$.
Solving for x, $x=\frac{7f-165}{3}=2f-55+\frac{f}{3}$

f	x	S	Dad's age when son born	
21	-6	12	9 - rejected	
24	1	9	15 - rejected	
27	8	6	$21 \Rightarrow$	**(8, 27, 6)**
30	15	3	$27 \Rightarrow$	**(15, 30, 3)**

3. $\frac{3900}{R}-0.8=\frac{3900}{R+100} \Leftrightarrow \frac{39000-8R}{10R}=\frac{3900}{R+100}$
Since $R \neq 0,-100$, cross multiplying gives us $\cancel{39000R}+3900000-8R^2-800R=\cancel{39000R}$
$\Leftrightarrow 8R^2+800R-3900000=0$
$\Leftrightarrow R^2+100R-(4875)100=0$
Noticing that $65\cdot 75=4875 \Rightarrow (R-650)(R+750)=0$, or, using the quadratic formula directly,
$$R=\frac{-100\pm\sqrt{100^2+4(4875)100}}{2}=\frac{-100\pm\sqrt{4\cdot 100(25+4875)}}{2}=\frac{-100\pm 2\cdot 10\cdot 70}{2}=-50\pm 700$$
$\Rightarrow R=650$
Thus, the required travel time $=\frac{7800}{750}=\frac{780}{75}=\frac{156}{15}=\underline{\mathbf{10.4}}$ hours.

Detailed Solutions for GBML Meet 1 - OCTOBER 2017

ROUND 3

1. $\sqrt{\dfrac{\sqrt[12]{64}\cdot\sqrt[8]{16}}{2^{-3}-3^{-2}}}=\sqrt{\dfrac{\sqrt[12]{2^6}\cdot\sqrt[8]{2^4}}{\frac{1}{8}-\frac{1}{9}}}=\sqrt{\dfrac{\sqrt{2}\cdot\sqrt{2}}{\frac{9-8}{72}}}=\sqrt{144}=\underline{\mathbf{12}}$

2. $\sqrt{2\sqrt{99}+\dfrac{2}{10+3\sqrt{11}}}=\sqrt{6\sqrt{11}+\dfrac{2}{10+3\sqrt{11}}\cdot\dfrac{10-3\sqrt{11}}{10-3\sqrt{11}}}=\sqrt{6\sqrt{11}+\dfrac{20-6\sqrt{11}}{100-99}}=\sqrt{20}=\underline{\mathbf{2\sqrt{5}}}$.

3. $\left(\sqrt[4]{4}+3\sqrt[6]{27}+\sqrt[6]{125}\right)\left(3\sqrt{3}-\sqrt{5}+\sqrt{2}\right)=\left(\sqrt{2}+3\sqrt{3}+\sqrt{5}\right)\left(3\sqrt{3}-\sqrt{5}+\sqrt{2}\right)$
$=\left(\left(3\sqrt{3}+\sqrt{2}\right)+\sqrt{5}\right)\left(\left(3\sqrt{3}+\sqrt{2}\right)-\sqrt{5}\right)=\left(3\sqrt{3}+\sqrt{2}\right)^2-\left(\sqrt{5}\right)^2$
$=27+2+6\sqrt{6}-5=\underline{\mathbf{24+6\sqrt{6}}}$ or, equivalently, $\underline{\mathbf{6\left(4+\sqrt{6}\right)}}$.

Detailed Solutions for GBML Meet 1 - OCTOBER 2017

ROUND 4

1. $12x^2 - 42 - 165x = 12x^2 - 165x - 42 = 3\left(4x^2 - 55x - 14\right)$
   Since the coefficient of the middle term is odd, we know that the lead coefficient 4 must factor as $4 \cdot 1$ and not $2 \cdot 2$ and we have $\underline{\mathbf{3(4x+1)(x-14)}}$.

2. $$\begin{aligned}&25x^2 + 105y - 49y^2 - 75x\\ &\Leftrightarrow \left(25x^2 - 49y^2\right) - (75x - 105y)\\ &\Leftrightarrow (5x+7y)(5x-7y) - 15(5x-7y)\\ &\Leftrightarrow \underline{\mathbf{(5x-7y)(5x+7y-15)}}\end{aligned}$$

3. $$\begin{aligned}&4a^2 + a^6 - a^4 - 4\\ &\Leftrightarrow \left(a^6 - a^4\right) + \left(4a^2 - 4\right)\\ &\Leftrightarrow a^4\left(a^2-1\right) + 4\left(a^2-1\right)\\ &\Leftrightarrow \left(a^2-1\right)\left(a^4+4\right)\\ &\Leftrightarrow (a+1)(a-1)\left(a^4+4\right)\end{aligned}$$
   … and we're done? Not Quite!
   Consider $\left(a^4+4\right)$ as $\left(a^4+4a^2+4\right)-4a^2$.
   $\Leftrightarrow \left(a^2+2\right)^2 - (2a)^2$ - the difference of perfect squares!!
   Thus, the complete factorization is $\underline{\mathbf{(a+1)(a-1)\left(a^2+2a+2\right)\left(a^2-2a+2\right)}}$.

Detailed Solutions for GBML Meet 1 - OCTOBER 2017

ROUND 5

1. Since adding multiples of 360° to any of the arguments does not change the value of the trig functions, we have

$$\tan(-855° + 1080°)\cos(-480° + 720°) + \sin(-660° + 720°)\cot(-930° + 1080°)$$

$$\Leftrightarrow \tan(225°)\cos(240°) + \sin(60°)\cot(150°)$$

$$\Leftrightarrow \tan(45°)\cdot -\cos(60°) + \sin(60°)\cdot -\cot(30°)$$

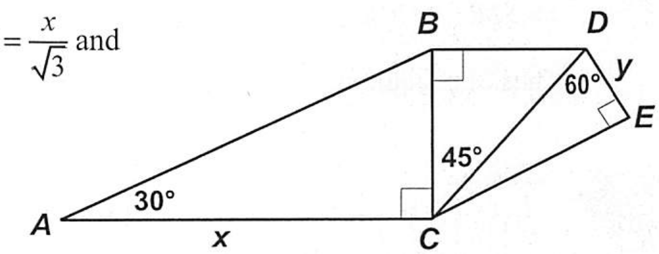
$$\Leftrightarrow 1\cdot -\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot -\sqrt{3} = -\frac{1}{2} - \frac{3}{2} = -2$$

$$\sec(x) = -2 \Leftrightarrow \cos(x) = -\frac{1}{2}$$

$$\Rightarrow x = \underline{\mathbf{120}}°,\ \underline{\mathbf{240}}°$$

2. In ΔABC, $BC = \frac{x}{\sqrt{3}}$. In ΔBCD, $BD = BC = \frac{x}{\sqrt{3}}$ and $DC = \frac{x}{\sqrt{3}}\cdot\sqrt{2}$. In ΔCDE,

$$y = \frac{DC}{2} = \frac{\frac{x}{\sqrt{3}}\cdot\sqrt{2}}{2} = \frac{\frac{x\sqrt{6}}{3}}{2} = \underline{\left(\frac{\sqrt{6}}{6}\right)x}.$$



3. 60 mph = 88 fps (feet per second) $\Rightarrow$ 45 mph = 66 fps.
If you are unfamiliar with this first equivalence, here's the derivation.

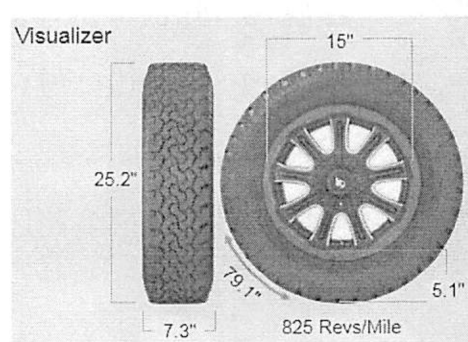
$$\left[\frac{\cancel{60}\ \cancel{\text{mi}}}{\cancel{\text{hr}}}\cdot\frac{5280\text{ft}}{\cancel{\text{mi}}}\cdot\frac{1\ \cancel{\text{hr}}}{\cancel{60}\ \cancel{\text{min}}}\cdot\frac{1\ \cancel{\text{min}}}{60\text{sec}} = \frac{528\cancel{0}\ \text{ft}}{6\cancel{0}\ \text{sec}} = 88\text{ fps}\right]$$

In one second, the automobile travels $66\cdot 12$ inches and the tire completes 10 revolutions which is equivalent to $10(\pi d)$.

Therefore, $d = \frac{66\cdot 12}{10\pi} = \frac{66\cdot 6}{5\pi} \Rightarrow (a,b) = \underline{(\mathbf{396,5})}$.

For comparison, $\frac{396}{5\pi} \approx 25.21$ inches.

The 185/70R15 has a diameter of 25.2 inches. It is used on the Porsche 911, the Volvo 164 and the Mercury Marquis.
185 is the width of the tire (in millimeters).
R means radial. 15 is the diameter of the rim.
70 is referred to as the aspect ratio, i.e. the ratio of the sidewall height (5.1") to the width of the tire (7.3").
25.4 mm = 1 inch. Have fun with the conversions!



Detailed Solutions for GBML Meet 1 - OCTOBER 2017

TEAM ROUND

1. The required ratio is $\dfrac{R+\frac{3}{2}}{R-\frac{3}{2}}$. Since $\left(R+\frac{3}{2}\right)T=6 \Rightarrow T=\dfrac{6}{R+1.5}$ and

$\left(R-\frac{3}{2}\right)(1.95-T)=6 \Rightarrow T=1.95-\dfrac{6}{R-1.5}$, by transitivity, $\dfrac{6}{R+1.5}+\dfrac{6}{R-1.5}=1.95$

$\dfrac{12}{2R+3}+\dfrac{12}{2R-3}=\dfrac{39}{20} \Leftrightarrow \dfrac{4}{2R+3}+\dfrac{4}{2R-3}=\dfrac{13}{20}$

Multiplying by the LCD $=20(2R+3)(2R-3)$,

$4\cdot 20(2R-3)+4\cdot 20(2R+3)=13(2R+3)(2R-3)$

$\Rightarrow 320R=13\left(4R^2-9\right)=52R^2-117$

$\Rightarrow 52R^2-320R-117=(2R-13)(26R+9)=0 \Rightarrow R=6.5$

Thus, the required ratio is $\dfrac{6.5+1.5}{6.5-1.5} \Rightarrow \underline{\mathbf{8:5}}$.

2. $\dfrac{x-a}{b}-\dfrac{b}{x-a}=\dfrac{a}{x-b}-\dfrac{x-b}{a} \Leftrightarrow \dfrac{(x-a)^2-b^2}{b(x-a)}=\dfrac{a^2-(x-b)^2}{a(x-b)}$

Cross multiplying, $\left((x-a)^2-b^2\right)\cdot a(x-b)=\left(a^2-(x-b)^2\right)\cdot b(x-a)$

Moving both terms to the left side of the equation, and factoring the differences of perfect squares, $(x-a+b)\underline{(x-a-b)}\cdot a(x-b)-(a+x-b)\underline{(a-x+b)}\cdot b(x-a)=0$

Notice that the underlined terms are opposites of each other. Factoring out this common factor,

$(x-a-b)\left((x-a+b)\cdot a(x-b)+(a+x-b)\cdot b(x-a)\right)=0$

$\Leftrightarrow (x-a-b)\cdot$

$\left(ax^2-a^2x+\cancel{abx}-\cancel{abx}+\cancel{a^2b}-\cancel{ab^2}\ +\ \cancel{abx}+bx^2-b^2x-\cancel{a^2b}-\cancel{abx}+\cancel{ab^2}\right)$

$\Leftrightarrow (x-a-b)\left(ax^2+bx^2-a^2x-b^2x\right)$

$\Leftrightarrow (x-a-b)\cdot x\cdot\left((a+b)x-\left(a^2+b^2\right)\right)$

$$\Rightarrow \begin{cases} x=a+b \\ x=0 \\ x=\dfrac{a^2+b^2}{a+b} \end{cases}$$

$a=1,2$ and $b=3,4 \Rightarrow a+b=4,5,6$, $a^2+b^2=10,\ 13,\ 17,\ 20$. Thus, integer solutions are $0,4,5,6$; the non-integer solutions are $\dfrac{10}{4},\dfrac{13,17}{5},\dfrac{20}{6}$. $(N,k)=\underline{(\mathbf{8,4})}$.

Detailed Solutions for GBML Meet 1 - OCTOBER 2017

TEAM ROUND - continued

3. We require that $\frac{xy}{x+y} = k$, where k is a natural number.

   Cross multiplying and solving for y in terms of x and k, $y = \frac{kx}{x-k}$.

   By long division, $y = k + \frac{k^2}{x-k}$; so, it is required that $x-k$ is a factor of k^2.

k	k^2	$x-k=$	$x=$	$y = k + \frac{k^2}{x-k}$	Ordered pairs (x, y), where $x < y$	Ct
3	9	1,3,9	4,6,12	12,6,4	(4,12)	1
4	16	1,2,4,8,16	5,6,8,12,20	20,12,8,6,5	(5,20), (6,12)	2
5	25	1,5,25	6,10,30	The y-values	(6,30)	1
6	36	1,2,3,4,6,9,12,18,36	7,8,9,10,12,15,18, 24,42	are always the x-values in	(7,42), (8,24), (9,18), (10,5)	4
7	49	1,7,49	8,14,56	reverse order!	(8,56)	1
8	64	1,2,4,8,16,32,64	9,10,12,16,24,40,72		(9,72), (10,40), (12,24)	3
9	81	1,9,27,81	10,18,36,90		(10,90), (18,36)	2
					Total:	**14**