# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 1 - OCTOBER 2018

**ROUND 1**- Arithmetic: Open

1. ______________________________

2. ______________________________

3. ( ____________ , ____________ )

1. Let $K$ be the median value in the set of odd composite numbers less than 32. Let $J$ be the smallest odd composite number greater than $10\left(2^3\right)+3^2$. Compute the *number* of factors of $P = K \cdot J$.

2. Let $x☺ = 8\left(x^2+7\right)$, $x☹ = 3\left(x^2+2\right)$ and $x \otimes y = \sqrt{x☺ \cdot y☹}$.
   Compute $11 \otimes 5$.

3. There are $P$ natural numbers between 1 and 275,inclusive,which are divisible by 3 or 5, but not 7.The number of even natural numbers that divide $P$ is $K$.
   Compute the ordered pair $(P, K)$.

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 1 - OCTOBER 2018

**ROUND 2**- Algebra 1: Word Problems

1. ______________________

2. ______________________

3. ( ______ , ______ , ______ )

1. A collection of 40 coins is worth \$6.95. The collection consists exclusively of nickels, dimes and quarters. There are twice as many quarters as dimes. Compute the total value (in cents) of the nickels.

2. Given a square $S$ whose perimeter is 48 inches. A square $T$ whose perimeter is $4x$ inches is drawn inside square $S$. Compute $x$ so that the area of the region inside $S$ and outside $T$ will equal $37\frac{1}{2}\%$ of the area of $S$.

3. My wife is 3 years older than I am.Our first son was born in the same year we were married. Our second son was born two years later. The sum of our current ages is 215. In three years, my youngest son will be half my age. It is now 2018. Compute the ordered triple $(f, s, y)$, where $f$ denotes my age;$s$, my son's age; and $y$, the year my wife and I were married.

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 1 - OCTOBER 2018

**ROUND 3**- Algebra 1: Exponents and Radicals

1. ______________________

2. ( __________ , __________ )

3. ______________________

1. If $2^{4x-3} - 3 = 5^3$, compute $4^{-x}$.

2. Compute $\left( \dfrac{6^{-1} - 9^{-\frac{1}{2}}}{27^{\frac{1}{3}} - 4^{-1}} \right)^{-1}$.

3. Solve over the reals. $\dfrac{\sqrt{2x-1}}{x} - \dfrac{1}{\sqrt{x+1}} = 0$

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 1 - OCTOBER 2018

**ROUND 4**- Algebra 2: Factoring

1. ______________________________

2. ______________________________

3. ______________________________

1. In each of the following years, there was (or will be) a presidential election:
   2012  1980  2020  1920  2000  1792  1812

   Determine the prime factorization of the median year in this list.
   Express in the form $p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_1} \cdot \ldots$ for as many prime factors as is necessary, where $p_1 < p_2 < p_3 < \ldots$.

2. Factor completely over the integers. $16x^{16} + 1 - 8x^8$

3. $p(x) = 2x^3 - 3x^2 - 32x + 48$ is to be factored over the integers.
   Let $f(x)$ denote a *linear*factor of $p(x)$. Compute the *largest* possible value of $|f(-3)|$.

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 1 - OCTOBER 2018

**ROUND 5**- Trig: Angular and Linear Velocity, Right Triangles

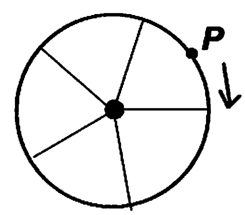
1. ______________________

2. ______________________

3. ______________________

1. In right$\Delta KEN$, $\overline{KN} \perp \overline{NE}$ and $\sin K = \frac{21}{29}$.
   Compute $\cos K$ .

2. A point $P$ on the rim of a wheel which is turning at a uniform rate moves through $3\frac{3}{4}^{\circ}$ in 0.001 seconds. Express its angular speed in revolutions per minute.




3. At $k$ minutes after 3:00, the minute hand has moved through $\frac{\pi}{5}$ radians from the top of the hour. $\theta$ is the radian measure of the angle formed by the minute and hour hands at this time. Compute the ordered pair $(k, \theta)$.

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 1 - OCTOBER 2018

### TEAM ROUND

3 pts. 1. ______________________

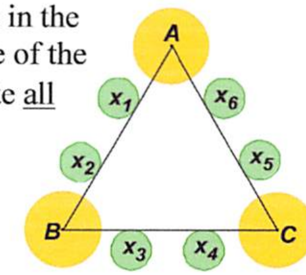
3 pts. 2. ______________________

4pts. 3.______________________________________________

1. The product of the digits of a 3-digit base 10 integer *N* is 48. If one such number is randomly chosen, compute the probability that it is *not* a multiple of 3.

2. Compute *all* real numbers for which $\sqrt{1+3\sqrt{x-1}}=\sqrt{\frac{x+1}{3}+1}$.

3. Using all the natural numbers 1 to 9, place numbers in the circles in the diagram at the right so that the sum of the *four* values on any side of the triangle is 20. If $A<B<C$, $x_1<x_2$, $x_3<x_4$, and $x_5<x_6$, compute all possible 9-tuples $(A,B,C,x_1,x_2,x_3,x_4,x_5,x_6)$.



**GREATER BOSTON MATHEMATICS LEAGUE**

**MEET 1 - OCTOBER 2018**

# Answer Sheet

Round 1

1. 12

2. 288

3. (110, 4)

Round 2

1. 35

2. $3\sqrt{10}$

3. (71, 36, 1982)

Round 3

1. $\frac{1}{32}$ (or 0.03125)

2. $-\frac{33}{2}$ or $-16.5$

3. $\frac{\sqrt{5}-1}{2}$

Round 4

1. $2^2 \cdot 3^2 \cdot 5 \cdot 11$

2. $\left((2x^4+1)(2x^4-1)\right)^2$ or $(2x^4+1)^2(2x^4-1)^2$

3. 9

Round 5

1. $\frac{20}{29}$

2. 625

3. $\left(6, \frac{19\pi}{60}\right)$

Team Round

1. $\frac{3}{7}$ (3 pts)

2. 2, 26 (4 pts)

3. (4, 5, 6, 2, 7, 1, 9, 3, 8)
   (3, 5, 7, 2, 6, 1, 9, 4, 8) (3 pts)

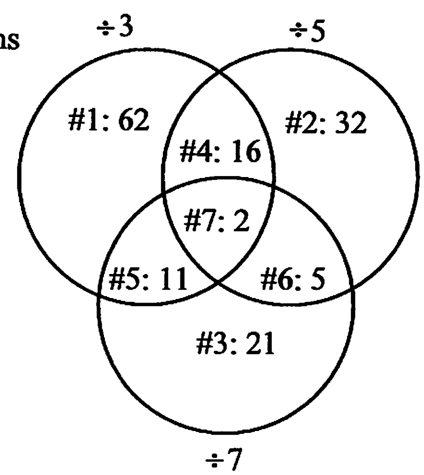
**Detailed Solutions for GBML Meet 1 - OCTOBER 2018**

## ROUND 1

1. $K$ is the median value in the set $S=\{9,15,21,25,27\}$. Thus, $K=21$.
   $10(2^3)+3^2=89$. The smallest composite number greater than 89 is $91=7(13)$.
   $P=21\cdot 91=3^1\cdot 7^2\cdot 13^1$ has $(1+1)(2+1)(1+1)=$ **<u>12</u>** factors.

2. Given: $x☺=8(x^2+7)$, $x☹=3(x^2+2)$ and $x\otimes y=\sqrt{x☺\cdot y☹}$
   $11\otimes 5=\sqrt{8(11^2+7)\cdot 3(5^2+2)}=\sqrt{8\cdot 128\cdot 3\cdot 27}=\sqrt{2^{10}\cdot 3^4}=2^5\cdot 3^2=32(9)=\mathbf{\underline{288}}$.

3. Using a Venn Diagram, we can easily divide the 275 integers between 1 and 275, inclusive, into non-overlapping sets with respect to their divisibility by 3, 5, and 7.
   For example, region #4 is home to integers divisible by both 3 and 5, but not by 7.
   Region #7 is home to integers are divisible by 3, by 5, and by 7, namely, 105 and 210.
   Multiples of 15 are in regions 4 or 7.
   Since there are 18 such integers $\left(\dfrac{275}{15}=18^+\right)$, region 4 contains 16 integers. Similar calculations provide the count for each region. Integers divisible by 3 or 5, but not by 7 are found in regions 1, 2, and 4, a total of $P=62+16+32=110$ integers.
   $110=2\cdot 5\cdot 11\Rightarrow$ 2, 10, 22, 110
   Thus, $(P,K)=\underline{(\mathbf{110,4})}$.



**Detailed Solutions for GBML Meet 1 - OCTOBER 2018**

**ROUND 2**

1. Let $N$, $D$, $Q$ denote the number of nickels, dimes, and quarters, respectively. Then:

$\begin{cases} N+D+Q=40 \\ 5N+10D+25Q=695 \\ Q=2D \end{cases}$. Dividing through by 5 and substituting, we have

$\begin{cases} N=40-3D \\ N+2D+5Q=139 \Rightarrow N+12D=139 \end{cases} \Rightarrow 40+9D=139 \Rightarrow D=11 \Rightarrow N=7 \Rightarrow \underline{\mathbf{35}}$.

2. The area of $S$ is $12^2=144$. The area of the region inside $S$ and outside $T$ is $67\tfrac{1}{2}\%=\frac{5}{8}$ of the area of $S$, i.e., $\frac{5}{8}\cdot 144=5\cdot 18=90 \Rightarrow x=\underline{\mathbf{3\sqrt{10}}}$.

3. Let $x$ and $y$ denote my age and my oldest son's age, respectively. Then:

$x+(x+3)+y+(y-2)=215 \Rightarrow x+y=107$.

$x+3=2(y-2+3) \Rightarrow x=2y-1$.

Substituting, $3y=108 \Rightarrow y=36,\ x=71, y=2018-36=1982$.

Therefore, $(f,s,y)=\underline{\mathbf{(71,36,1982)}}$.

**Detailed Solutions for GBML Meet 1 - OCTOBER 2018**

**ROUND 3**

1. $2^{4x-3}-3=5^3 \Rightarrow 2^{4x-3}=128=2^7 \Rightarrow 4x-3=7 \Rightarrow x=\frac{5}{2}$.

   $4^{-\frac{5}{2}}=\left(4^{\frac{1}{2}}\right)^{-5}=2^{-5}=\underline{\mathbf{\frac{1}{32}}}$ (or $\underline{\mathbf{0.03125}}$).

2. $\left(\frac{6^{-1}-9^{-\frac{1}{2}}}{27^{\frac{1}{3}}-4^{-1}}\right)^{-1}=\left(\frac{3-\frac{1}{4}}{\frac{1}{6}-\frac{1}{3}}\right)\cdot 12=\frac{36-3}{2-4}=\frac{33}{-2}=\underline{\mathbf{-\frac{33}{2}}}$ or $\underline{\mathbf{-16.5}}$.

3. $\frac{\sqrt{2x-1}}{x}-\frac{1}{\sqrt{x+1}}=0 \Leftrightarrow \frac{\sqrt{2x-1}}{x}=\frac{1}{\sqrt{x+1}}$.

   Cross-multiplying, $\sqrt{2x-1}\sqrt{x+1}=x \Leftrightarrow \sqrt{2x^2+x-1}=x$.

   Squaring both sides, $2x^2+x-1=x^2 \Leftrightarrow x^2+x-1=0$.

   Applying the quadratic formula, $x=\frac{-1\pm\sqrt{5}}{2}$.

   $x=\frac{-1-\sqrt{5}}{2}$ is rejected, because this value causes both radicands in the original equation to be negative, and we are required to solve over the real numbers.

   Thus, the solution is unique, namely, $\underline{\mathbf{\frac{\sqrt{5}-1}{2}}}$.

**Detailed Solutions for GBML Meet 1 - OCTOBER 2018**

**ROUND 4**

1. The median value in $\{1792, 1812, 1920, 1980, 2000, 2012, 2020\}$ is 1980.
   $1980 = 4(495) = 4(5)(99) = \underline{\mathbf{2^2 \cdot 3^2 \cdot 5 \cdot 11}}$.

2. $16x^{16} - 8x^8 + 1 = (4x^8 - 1)^2$
   As the difference of perfect squares, we have $\underline{\mathbf{((2x^4+1)(2x^4-1))^2}}$ or $\underline{\mathbf{(2x^4+1)^2(2x^4-1)^2}}$.

3. Using synthetic substitution, we experiment with possible roots until we confirm that $x = 4$ is a root. Additionally, we have a partial factorization, namely, $(x-4)(2x^2+5x-12)$
   $\Rightarrow (x-4)(x+4)(2x-3)$

| | | | | |
|---|---|---|---|---|
| | 2 | −3 | −32 | 48 |
| 4\| | 2 | 5 | −12 | 0 |

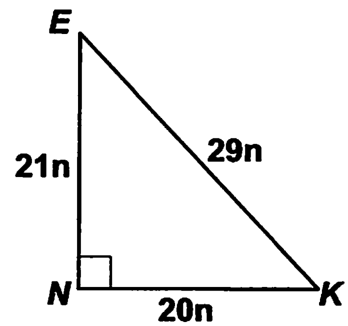
   Evaluating each factor for $x = -3$, and taking the absolute value, we have 7, 1, **9**.

   Alternately, $p(x)$ may be factored by grouping and extracting a common binomial factor.

   $2x^3 - 3x^2 - 32x + 48 = x^2(2x-3) - 16(2x-3) = (2x-3)(x^2-16)$, and the result follows.

**Detailed Solutions for GBML Meet 1 - OCTOBER 2018**

**ROUND 5**




1. Given: $\sin K = \frac{EN}{EK} = \frac{21}{29}$

   Using the Pythagorean Triple $(20, 21, 29)$, we know the sides of $\Delta KEN$ are multiples of this triple. This result is easily confirmed by the Pythagorean Theorem. $x^2 + 21^2 = 29^2 \Rightarrow x^2 = 29^2 - 21^2$

   Recognizing the difference of perfect squares $a^2 - b^2 = (a+b)(a-b)$, we can avoid any "nasty" squaring.

   $29^2 - 21^2 = (29+21)(29-21) = 50 \cdot 8 = 25 \cdot 16 \Rightarrow x = 5 \cdot 4 = 20$.

   Thus, $\cos K = \underline{\mathbf{\frac{20}{29}}}$.

   Alternately, appealing to $\sin^2\theta + \cos^2\theta = 1$, we get the same answer.

2. $\frac{3.75°}{0.001 \text{ sec}} = \frac{\frac{3.75}{360}}{\frac{0.001}{60}} \text{ RPM} = \frac{3.75}{6(.001)} = \frac{3750}{6} = \underline{\mathbf{625}} \text{ RPM}$.

3. $\frac{\pi}{5}$ radians is equivalent to $\frac{\pi}{5} \cdot \frac{180}{\pi} = 36°$. The minute hand moves through 360° in one hour, or 6° per minute. Therefore, $k = 6$. Since the minute hand moves 12 times faster than the hour hand, the hour hand has advanced 3°. Thus, the angle between the minute and hour hand is $(90 - 36 + 3) = 57° \cdot \frac{\pi}{180} = \frac{19\pi}{60}$ radians. Thus, $(k, \theta) = \underline{\mathbf{\left(6, \frac{19\pi}{60}\right)}}$.

## Detailed Solutions for GBML Meet 1 - OCTOBER 2018

### TEAM ROUND

| Digits | # N-values | ÷3 |
|---|---|---|
| 1,6,8 | 6 | Yes |
| 2,4,6 | 6 | Yes |
| 2,3,8 | 6 | No |
| 3,4,4 | 3 | No |

1. The possible digits of $N$ are summarized in the chart at the right.Thus, the required probability is $\frac{9}{21}=\underline{\mathbf{\frac{3}{7}}}$.

2. Squaring, $\sqrt{1+3\sqrt{x-1}}=\sqrt{\frac{x+1}{3}}+1 \Rightarrow \cancel{1}+3\sqrt{x-1}=\frac{x+1}{3}+\cancel{1}+2\sqrt{\frac{x+1}{3}}$.

   Let $y=\sqrt{\frac{x+1}{3}}$. Then: $x=3y^2-1$. Substituting and squaring both sides,

   $3\sqrt{3y^2-2}=y^2+2y \Rightarrow 9\left(3y^2-2\right)=y^4+4y^3+4y^2 \Rightarrow y^4+4y^3-23y^2+18=0$

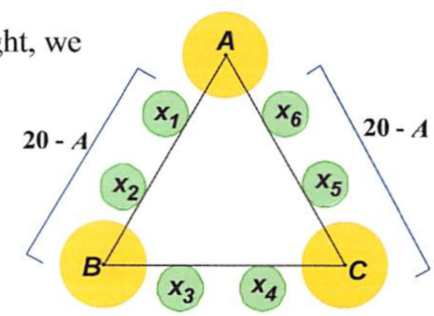
   $$\begin{array}{r|rrrrr} & 1 & 4 & -23 & 0 & 18 \\ \hline 1 & 1 & 5 & -18 & -18 & 0 \\ 3 & 1 & 8 & 6 & 0 & \end{array}$$

   Using synthetic substitution, $y=1,3 \Rightarrow x=\underline{\mathbf{2,\ 26}}$.

   Using the quadratic formula on the remaining trinomial,

   $y^2+8y+6=0 \Rightarrow y=\frac{-8\pm\sqrt{64-24}}{2}=-4\pm\sqrt{10}$. However, both $y$-values are negative and, therefore, extraneous.

3. Since $1+2+3+...+9=\frac{9\cdot 10}{2}=45$, from the diagram at the right, we have $2(20-A)+A+x_3+x_4=45 \Rightarrow x_3+x_4=A+5$




   By symmetry, $\begin{array}{l} x_5+x_6=B+5 \\ x_1+x_2=C+5 \end{array}$.

   Thus, $x_1+...+x_6=A+B+C+15$. Adding $A+B+C$ to both sides, we have $45=2(A+B+C)+15 \Rightarrow \boxed{A+B+C=15}$

   There are only 4 possible ordered triples $(A,B,C)$ for which $A+B+C=15$ and $A<B<C$. In each case, the following chart organizes the corresponding values which must be assigned to the $x$'s. $x_3+x_4$ is the sum of the pair of numbers opposite $A$, and, similarly, for $B$ and $C$.

| **($A$, $B$, $C$)** | **($x_3+x_4$, $x_5+x_6$, $x_1+x_2$)** | **$x$-values** | **Pairings: ($x_3$,$x_4$), ($x_5$,$x_6$), ($x_1$,$x_2$)** |
|---|---|---|---|
| (4, 5, 6) | (9, 10, 11) | 1, 2, 3, 7, 8, 9 | (2, 7), (1,9), (3,8) |
| (3, 5, 7) | (8, 10, 12) | 1, 2, 4, 6, 8, 9 | (2,6), (1,9), (4,8) |
| (3, 4, 8) | (8, 9, 13) | 1, 2, 5, 6, 7, 9 | None |
| (2, 4, 9) | (7, 9, 14) | 1, 3, 5, 6, 7, 8 | None |

Thus, the ordered 9-tuples are **<u>(4,5,6,2,7,1,9,3,8), (3,5,7,2,6,1,9,4,8)</u>**.