GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2014

ROUND 1 – Arithmetic - Open

1. ______________________

2. (_____ , _____ , _____)

3. ______________________

1. How many integers between 1 and 300 exclusive are divisible by neither 3 nor 5?

2. The year my son was born, there were 4 Thursdays, 4 Saturdays, and 4 Sundays in December. Let (Thu, Sat, Sun) denote the number of Thursdays, Saturdays, and Sundays respectively during January of the following year.
Compute the ordered triple (Thu, Sat, Sun).

3. Point N lies on $\overline{PQ}$, dividing $\overline{PQ}$ into two segments, one of whose lengths is seven times that of the other. If the coordinate of P is $\frac{1}{9}$ and the coordinate of Q is $\frac{1}{5}$, compute <u>all</u> possible coordinates of point N as reduced fractions.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2014

ROUND 2 – Simultaneous Linear equations, Word Problems, Matrices

1. ______________________

2. ______________________

3. ______________________

1. Given: x and k are positive integers and $\begin{vmatrix} x & 4 \\ -1 & x \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & k \end{vmatrix}$
   Determine minimum value of k, if $k + x > 10$.

2. The sum of the ages of A, B, and C is 105 years.
   The ratio of A's age to B's age is 7 : 10, while the ratio of B's age to C's age is 5 : 9.
   In how many years will the ratio of C's age to A's age be 20 : 9?

3. Given the system of equations $\begin{cases} 11x + 7y = 5 \\ 13x - 9y = -20 \end{cases}$
   Compute the product xy.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2014

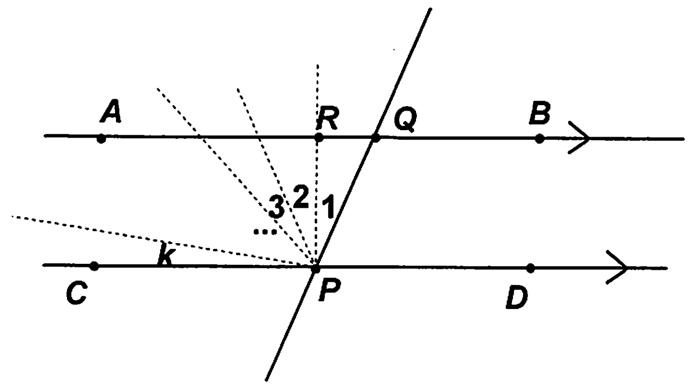
ROUND 3 – Geometry - Angles and Triangles

1. ______________________

2. (__________ , __________)

3. ________°, ________°, ________°

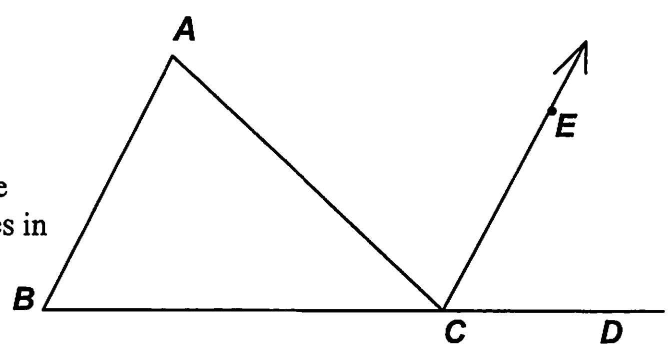
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.



1. Two parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are cut by transversal $\overleftrightarrow{PQ}$. A pair of adjacent angles formed by $\overleftrightarrow{PQ}$ and $\overleftrightarrow{CD}$ have measures in a 7 : 5 ratio. The obtuse angle is subdivided into k congruent angles so that $\overline{PR} \perp \overline{CD}$. Determine the value of k.

2. The measures of the interior angles of a triangle are in a 2 : 3 : 4 ratio. If the smallest is decreased by $k°$ and the largest is increased by $k°$, the measure of the angles (in the same order) is 3 : 5 : n. Compute the ordered pair (k,n).

3. In the diagram at the right, $\overrightarrow{CE}$ bisects angle ACD, $\overrightarrow{CE} \parallel \overline{AB}$, $m\angle ACE : m\angle ACB = a : b$, where a and b are positive integers and $a+b=10$. If all angles in ΔABC have integer measures, compute the three possible degree-measures of $\angle B$.



Note that no single diagram can be representative of all three possibilities.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2014

ROUND 4 – Algebra 2 – Quadratic Equations, Problems involving Them and Theory of Quadratics

1. (____________ , ____________)

2. ______________________________

3. $f(2n) =$ ____________________

1. The solution set for the equation $4M^2 + 3MN - 8M - 6N = 0$ is $M = a$ and $M = bN$. Compute the ordered pair (a, b).

2. Determine <u>all</u> possible values of the constant m for which the quadratic equation, $mx^2 + 3mx + 2m = x^2 + 3x + 1$ has exactly one solution.

3. The roots of the equation $3x^2 - 4x - 5 = 0$ are r_1 and r_2. The roots of $ax^2 + bx + c = 0$ are $2r_1 + 1$ and $2r_2 + 1$, where a, b and c are integers and $\underline{a < 0}$. Compute the ordered triple (a, b, c) for which the product abc is a maximum.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2014

ROUND 5 – Trig Equations

1. ______________________

2. ______________________

3. ______________________

1. Given: $\sin\theta = 2\cos\theta$
   Compute $\tan\left(\frac{\pi}{2}+\theta\right)$.

2. Given: $\sin(30°+11°)+\sin(30°-11°)=\sin\theta$
   Compute <u>all</u> values of θ over the interval $0° \le \theta < 360°$.

3. For what value(s) of x over $0° \le x < 360°$ is the following statement true:

$$\frac{\sin 20°\cos 25° - \sin 155°\cos 160°}{\sin 185°\sin 220° - \cos 320°\cos 355°} = \cos(270°+x)$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2014

TEAM ROUND

3 pts. 1. ______________________

3 pts. 2. ______________________

4 pts. 3. $A =$ ______

____

1. P is a prime number, all of whose digits are prime. P can be written as the sum of three distinct primes in at least 5 different ways. Find the smallest possible value of P.

2. Compute all ordered pairs of positive real numbers (x, y) which satisfy the following system of equations:

$$\begin{cases} \sqrt[3]{\dfrac{x+y}{x-y}} + 2\left(\sqrt[3]{\dfrac{x-y}{x+y}}\right) = 3 \\ x+y=5 \end{cases}$$

3. Two angles of ΔPQR measure $(3x + 1)°$ and $(2x + A)°$, where x and A are positive integers. For a unique integer A, ΔPQR is isosceles, for any choice of vertex angle. Let V denote the vertex angle of ΔPQR and B denote a base angle. Compute A and all possible ordered pairs (V, B).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2014

Answer Sheet

Round 1

1. 159
2. (5, 5, 4)
3. $\frac{11}{90}, \frac{17}{90}$

Round 2

1. 8
2. 6
3. $-\frac{3}{4}$

Round 3

1. 7
2. (4, 7)
3. 30°, 60°, 80° (in any order)

Round 4

1. $\left(2, -\frac{3}{4}\right)$
2. 5
3. $(-3, 14, 9)$

Round 5

1. $-\frac{1}{2}$
2. 79°, 101°
3. 270°

Team Round

1. 37 (3 pts)
2. $\left(\frac{45}{16}, \frac{35}{16}\right)$ (3 pts)
3. $A = 2$
   (172, 4), (46, 67), (76, 52) (4 pts)

Detailed Solutions for GBML Meet 2 – NOVEMBER 2014

ROUND 1

1. There are 99 multiples of 3 in the specified interval, namely $3\cdot 1 \ldots 3\cdot(100-1)$.
   There are 59 multiples of 5 in the specified interval, namely $5\cdot 1 \ldots 5\cdot(60-1)$.
   However, there is an overlap, since multiples of 15 have been counted twice.
   There are 19 multiples of 15 ($1\cdot 5 \ldots 15(20-1)$).
   Therefore, there are $99+59-19=\underline{\mathbf{139}}$ integers divisible by 3 and/or 5 and, therefore, $298-139=\underline{\mathbf{159}}$ which are divisible by neither 3 nor 5.

2.

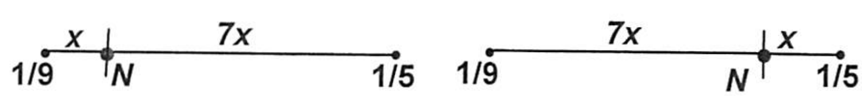
<u>**S**</u>	<u>**M**</u>	<u>**T**</u>	<u>**W**</u>	<u>**R**</u>	<u>**F**</u>	<u>**S**</u>
X				X		X
X				X		X
X				X		X
X				X		X

<u>**S**</u>	<u>**M**</u>	<u>**T**</u>	<u>**W**</u>	<u>**R**</u>	<u>**F**</u>	<u>**S**</u>
	1			X		X
X	8			X		X
X	15			X		X
X	22			X		X
X	29	30	31			

If the first of the month were on a Sunday, since there are 31 days in December, there would be 3 days in the last row of the calendar. This cannot be the case, since this would result in a 5th Sunday. The only way to eliminate another X in the Sunday, Thursday and Saturday column is for Dec 1st to fall on a Monday. Thus, January 1st in the new year fell on a Thursday and the January calendar looked like this:

<u>**S**</u>	<u>**M**</u>	<u>**T**</u>	<u>**W**</u>	<u>**R**</u>	<u>**F**</u>	<u>**S**</u>
				1		X
X				8		X
X				15		X
X				22		X
X				29	30	31

and there are <u>**5**</u> Thursdays, <u>**5**</u> Saturdays and <u>**4**</u> Sundays.



3. There are 2 possibilities, depending on whether N is closer to $\frac{1}{9}$ or $\frac{1}{5}$.

Case 1:

$$\left(\frac{1}{5}-N\right)=7\left(N-\frac{1}{9}\right)\Rightarrow 9-45N=7\cdot 45N-7\cdot 5\Rightarrow 9-45N=315N-35\Rightarrow N=\frac{44}{360}=\frac{11}{90}$$

Case 2:

$$\left(N-\frac{1}{9}\right)=7\left(\frac{1}{5}-N\right)\Rightarrow 45N-5=63-315N\Rightarrow N=\frac{68}{360}=\frac{17}{90}$$

Therefore, the possible coordinates of N are $\underline{\mathbf{\frac{11}{90}}}$ or $\underline{\mathbf{\frac{17}{90}}}$.

Detailed Solutions for GBML Meet 2 – NOVEMBER 2014

ROUND 2

1. $\begin{vmatrix} x & 4 \\ -1 & x \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & k \end{vmatrix} \Rightarrow x^2+4=2k-3 \Rightarrow x^2=2k-7$

   $k=4 \Rightarrow x=1$, but $k+x<10$
   For $k=5,6,7$, x is not an integer.
   $k=8 \Rightarrow x=3$ and $k+x>10$
   Thus, the minimum value of k is **8**.

2. $\begin{cases} A+B+C=105 \\ A:B=7:10 \\ B:C=5:9 \end{cases} \Rightarrow \begin{cases} B=\frac{10}{7}A \\ C=\frac{9}{5}B=\frac{18}{7}A \end{cases}$

   Substituting, $A+\frac{10}{7}A+\frac{18}{7}A=105 \Rightarrow 5A=105 \Rightarrow A=21,\ B=30,\ C=54$

   Thus, $\frac{54+x}{21+x}=\frac{20}{9} \Rightarrow 420+20x=486+9x \Rightarrow 11x=66 \Rightarrow x=\underline{6}$.

   Alternately, the ratio of $A : C$ can be found by making the given ratios commensurate, i.e. by making comparisons to the same number.
   $\begin{cases} A:B=7:10 \\ B:C=5:9 \end{cases} \Leftrightarrow \begin{cases} A:B=7:10 \\ B:C=10:18 \end{cases} \Rightarrow A:C=7:18$ and we have the same result as above.

3. Given: (1) $11x+7y=5$
   (2) $13x-9y=-20$

   Adding, we have $24x-2y=-15 \Rightarrow$ (3) $x=\frac{2y-15}{24}$.

   Subtracting, we have $2x-16y=-25 \Rightarrow$ (4) $x=\frac{16y-25}{2}$

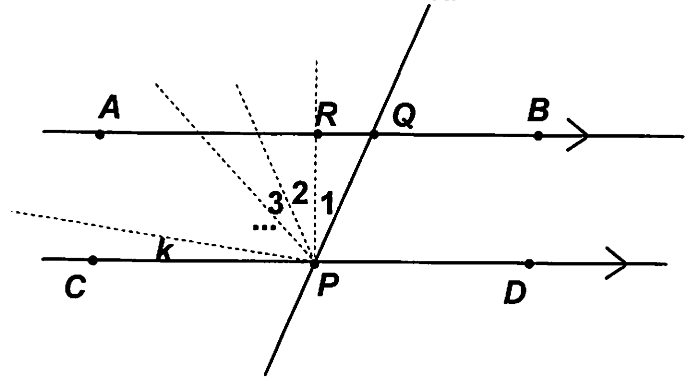
   Equating these x-expressions,
   $\frac{2y-15}{\cancel{24}^{12}}=\frac{16y-25}{\cancel{2}} \Rightarrow 2y-15=192y-300 \Rightarrow y=\frac{285}{190}=\frac{5\cdot 57}{5\cdot 38}=\frac{3}{2}$

   Substituting for y in (3), we have $x=\frac{3-15}{24}=-\frac{1}{2}$

   Thus, $xy=\underline{-\frac{3}{4}}$.

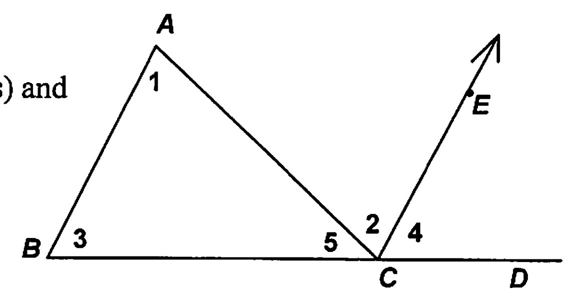
Detailed Solutions for GBML Meet 2 – NOVEMBER 2014

ROUND 3



1. $5x+7x=180 \Rightarrow x=15 \Rightarrow m\angle QPC=105°, m\angle QPD=75°$.
$m\angle RPD = 75+\frac{105}{k}=90 \Rightarrow \frac{105}{k}=15 \Rightarrow k=\underline{7}$.

2. Since $2x+3x+4x=180 \Rightarrow x=20$, the angles in the original triangle had measures of 40°, 60° and 80°. $\frac{40-k}{\cancel{60}^{12}}=\frac{3}{\cancel{5}^{1}} \Rightarrow 40-k=36 \Rightarrow k=4$ and the new triangle has angle measures of 36°, 60° and 84°. $(36,60,84)=12(3,5,7) \Rightarrow (k,n)=\underline{(4,7)}$.



3. Since $\overrightarrow{CE} \parallel \overline{AB}$, $m\angle 1 = m\angle 2$ (alt. interior angles) and $m\angle 3 = m\angle 4$ (corres. angles of parallels).
Since $\overrightarrow{CE}$ bisects angle ACD,
$m\angle 1 = m\angle 2 = m\angle 3 = m\angle 4$.
Let $(m\angle 2, m\angle 5)=(an, bn)$. Then:
$2an+bn=180$ and $a+b=10$
$(a,b)=$
$(9,1) \Rightarrow 19n=180$ rejected
$(8,2) \Rightarrow 18n=180 \Rightarrow n=10 \Rightarrow m\angle B = \underline{\mathbf{80}}°$
$(7,3),(6,4),(4,6),(3,7),(1,9)$ fail
$(5,5) \Rightarrow 15n=180 \Rightarrow n=12 \Rightarrow m\angle B = \underline{\mathbf{60}}°$
$(2,8) \Rightarrow 12n=180 \Rightarrow n=15 \Rightarrow m\angle B = \underline{\mathbf{30}}°$

Detailed Solutions for GBML Meet 2 – NOVEMBER 2014

ROUND 4

1. $4M^2 + 3MN - 8M - 6N = 0$
$\Leftrightarrow M(4M + 3N) - 2(4M + 3N) = 0$
$\Leftrightarrow (M-2)(4M+3N) = 0$
$\Leftrightarrow M = 2,\ M = -\frac{3}{4}N$
Thus, $(a,b) = \underline{\left(\mathbf{2}, -\frac{\mathbf{3}}{\mathbf{4}}\right)}$.

2. $mx^2 + 3mx + 2m = x^2 + 3x + 1$
Rearrange the terms to form a quadratic equation in x.
$(m-1)x^2 + 3(m-1)x + (2m-1) = 0$
One solution means the discriminant must be zero. $9(m-1)^2 - 4(m-1)(2m-1) = 0$.
$\Leftrightarrow 9m^2 - 18m + 9 - 8m^2 + 12m - 4 = m^2 - 6m + 5 = (m-5)(m-1) = 0$
$\Rightarrow m = \not{1}, \underline{5}$ [1 is extraneous, since the given equation degenerates to $2 = 1$.]

3. Given $3x^2 - 4x - 5 = 0$, from the root-coefficient relationship, without solving for the specific roots, we know that $r_1 + r_2 = +\frac{4}{3}$ and $r_1 r_2 = -\frac{5}{3}$.
Computing the sum and product of the roots of the new equation, we have

$$(2r_1 + 1) + (2r_2 + 1) = 2(r_1 + r_2) + 2 = 2\left(\frac{4}{3}\right) + 2 = \frac{14}{3}$$
$$(2r_1 + 1)\cdot(2r_2 + 1) = 4r_1 r_2 + 2(r_1 + r_2) + 1 = 4\left(-\frac{5}{3}\right) + 2\left(\frac{4}{3}\right) + 1 = \frac{-20 + 8 + 3}{3} = -3$$

Thus, the new equation is $x^2 - \frac{14}{3}x - 3 = 0$.
Eliminating fractions and maintaining a negative lead coefficient, we have $-3x^2 + 14x + 9 = 0$.
Since the product abc is always negative, a maximum value is obtained when $\gcd(a,b,c) = 1$.
Therefore, $(a,b,c) = \underline{(\mathbf{-3, 14, 9})}$.

Detailed Solutions for GBML Meet 2 – NOVEMBER 2014

ROUND 5

1. Provided $\cos\theta \neq 0$, dividing by $\cos\theta$, we have $\sin\theta = 2\cos\theta \Rightarrow \tan\theta = 2$
$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta = -\frac{1}{\tan\theta} = \underline{\mathbf{-\frac{1}{2}}}$.

2. Expanding $\sin(A+B)+\sin(A-B)$,
$\sin(30°+11°)+\sin(30°-11°) =$
$\left(\sin 30°\cos 11° + \cancel{\sin 11°\cos 30°}\right) + \left(\sin 30°\cos 11° - \cancel{\sin 11°\cos 30°}\right) = 2\sin 30°\cos 11°$
$2\sin 30°\cos 11° = \sin\theta \Rightarrow \cos 11° = \sin\theta \Rightarrow \sin 79° = \sin\theta$
Since $\sin 79°$ is a positive value, there are solutions in both quadrants 1 and 2.
The solutions are $\underline{\mathbf{79}}°$ and $(180° - 79°) =$ $\underline{\mathbf{101}}°$.

3. Given: $\frac{\sin 20°\cos 25° - \sin 155°\cos 160°}{\sin 185°\sin 220° - \cos 320°\cos 325°} = \cos(270°+x)$

Converting all values on the left side of the equation to reference values,

$$\frac{\sin 20°\cos 25° - \sin 155°\cos 160°}{\sin 185°\sin 220° - \cos 320°\cos 355°} = \frac{\sin 20°\cos 25° - \sin 25°(-\cos 20°)}{(-\sin 5°)(-\sin 40°) - (\cos 40°)(\cos 5°)}$$

$$\frac{\sin 20°\cos 25° + \sin 25°(\cos 20°)}{(\sin 5°)(\sin 40°) - (\cos 40°)(\cos 5°)} = \frac{\sin(20°+25°)}{-\cos(40°+5°)} = -\tan 45° = -1$$

$\cos(270°+x) = \sin x = -1 \Rightarrow x = \underline{\mathbf{270}}°$.

Detailed Solutions for GBML Meet 2 – NOVEMBER 2014

TEAM ROUND

1. If one of the primes were 2, then, since the other two primes would be odd, the sum P would be even and, therefore, not prime. With that in mind, we only consider three odd primes. Since P must have prime digits, our search starts with 23.

$23=\begin{cases}3+20\\5+18\end{cases}\Rightarrow\begin{cases}3+7+13\\5+7+11\end{cases}$ No other decompositions of 18 or 20 are possible, since the primes must be distinct (5 + 5 + 13 should not be considered).

The next victim is 37 and the decomposition is successful.

$$\underline{37}=\begin{cases}3+34\Rightarrow\begin{cases}3+(5+29)\\3+(11+23)\end{cases}\\5+32\Rightarrow\{5+(13+19)\\7+30\Rightarrow\begin{cases}7+(11+19)\\7+(13+17)\end{cases}\end{cases}$$

2. Given: $\begin{cases}\sqrt[3]{\dfrac{x+y}{x-y}}+2\left(\sqrt[3]{\dfrac{x-y}{x+y}}\right)=3\\x+y=5\end{cases}$

Let $A=\dfrac{x+y}{x-y}=\dfrac{x+(5-x)}{x-(5-x)}=\dfrac{5}{2x-5}$ and $B=\sqrt[3]{A}$ or $A=B^3$.

Then:

$$\sqrt[3]{\frac{x+y}{x-y}}+2\left(\sqrt[3]{\frac{x-y}{x+y}}\right)=3\Leftrightarrow B+2\left(\frac{1}{B}\right)=3\Rightarrow B^2-3B+2=0\Leftrightarrow(B-1)(B-2)=0\Rightarrow B=1,2$$

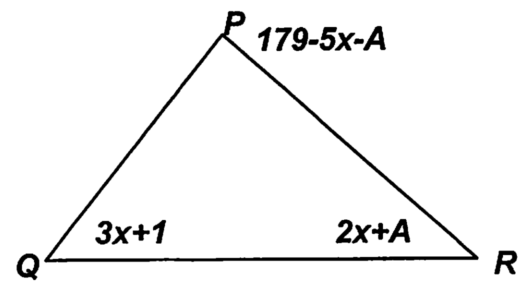
$\dfrac{5}{2x-5}=\begin{cases}1^3=1\\2^3=8\end{cases}$. Cross multiplying, $2x-5=\begin{cases}5\\\dfrac{5}{8}\end{cases}$

$2x-5=5\Rightarrow x=5\Rightarrow y=0$ and this case is rejected.

$2x-5=\dfrac{5}{8}\Rightarrow x=\dfrac{45}{16}$ and we have the unique solution $(x,y)=\underline{\left(\mathbf{\dfrac{45}{16},\dfrac{35}{16}}\right)}$.

Detailed Solutions for GBML Meet 2 – NOVEMBER 2014

TEAM ROUND



3. Case 1: $Q = R \Rightarrow 3x + 1 = 2x + A \Rightarrow x = A - 1$

Case 2: $P = Q \Rightarrow 3x + 1 = 179 - 5x - A$

$\Rightarrow x = \dfrac{178 - A}{8} = 22 + \boxed{\dfrac{2 - A}{8}}$

Case 3: $P = R \Rightarrow 2x + A = 179 - 5x - A$

$\Rightarrow x = \dfrac{179 - 2A}{7} = 25 + \boxed{\dfrac{4 - 2A}{7}}$

For $A = 2$, both boxed fractional expressions evaluate to integers.

For $A = 2$

$\Rightarrow$ (in case 1) $x = 1$ and $\angle$s of 4°, 4° and 172°

$\Rightarrow$ (in case 2) $x = 22$ and $\angle$s of 67°, 46° and 67°

$\Rightarrow$ (in case 3) $x = 25$ and $\angle$s of 76°, 52° and 52°

The next positive values for which this occurs is $A = 2 + 56k = 58, 114, \ldots$.

$\Rightarrow$ (in case 1) $x = 57, 113, \ldots$ and both m$\angle Q$ and m$\angle R$ would be greater than 90°.

Thus, 2 is the unique value of A.

$\Rightarrow$ <u>**$A = 2$ (172, 4), (46, 67), (76, 52)**</u>