GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2015

ROUND 1 – Arithmetic: Open

1. ______________________

2. ______________________

3. ______________________

1. Compute the number of odd factors of the number 324000.

2. At a local high school store, a particular marking pen was priced at 60 cents.
This price did not attract many buyers, and the stock of fewer than 200 pens remained unsold.
When the store reduced the price to a reasonable level, the entire remaining stock was sold for $29.87. By how much (in cents) was the price of a pen reduced?

3. Compute all ordered pairs (A,B), where A and B are base 10 integers, for which

$$A3B_{\text{base }9} - B3A_{\text{base }7} = 166_{\text{base }8}$$

Each of these terms denotes a 3-digit number in the specified base.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2015

ROUND 2 - Simultaneous Linear equations, Word Problems, Matrices

1. ______________________

2. (__________ , __________)

3. (_____ , _____ , _____ , _____)

1. Find all values for which $\begin{vmatrix} x & 3 \\ 4 & x-4 \end{vmatrix} = 0$ and $\begin{vmatrix} -3 & x-8 \\ x & 4 \end{vmatrix} \neq 0$.

2. Given a rectangle whose length is ℓ inches and whose width is w inches. If $2\frac{1}{2}$ inches are added to its length and $\frac{2}{3}$ inch is subtracted from its width, the area remains the same. If $2\frac{1}{2}$ inches are subtracted from its original length and $\frac{4}{3}$ inches are added to its original width, the area is again the same. Compute the ordered pair (l, w) .

3. Compute the ordered quadruple (A, B, C, D) which satisfy the following system of for equations:

$$\begin{cases} A+2B+C=6 \\ 2B+C+D=2 \\ C+D+A=9 \\ D+A+2B=-2 \end{cases}$$

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 3 – Geometry: Angles and Triangles

1. ____________________

2. $r =$ ____________________

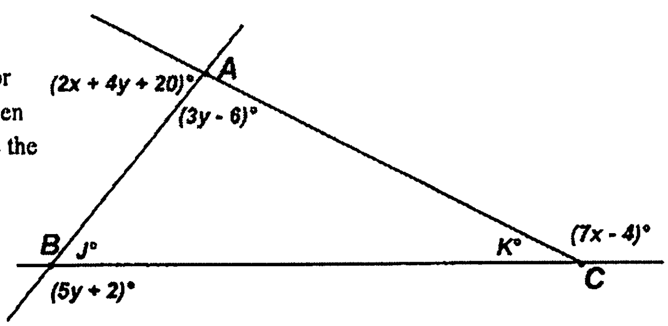
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. Given: $m(\measuredangle\alpha):m(\measuredangle\theta)=6:11$
   If 22° is added to half the supplement of the complement of α, the result equals twice the complement of the supplement of θ. Compute $m(\measuredangle\alpha)+m(\measuredangle\theta)$.

2. The area of ΔABC is numerically equal to n times the perimeter of ΔABC.
   A circle is inscribed in ΔABC. Express r, the radius of the inscribed circle, in terms of n.

3. The measures of the indicated interior and exterior angles of ΔABC are given in the diagram at the right. Compute the ordered quadruple (x, y, J, K).



GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2015

ROUND 4 - Algebra 2: Quadratic Equations and the Theory of Quadratics

1. ______________________

2. ______________________

3. ______________________

1. The area of a rectangle whose length is 1 less than twice its width is 21 units^2. Compute the perimeter of this rectangle.

2. If p and q are the nonzero roots of $x^2 + px + q = 0$, compute $p + q$.

3. The sequence $x+5,\ y-2,\ 5-3x$ is an arithmetic sequence.
   The sequence $3-x,\ y+2,\ 9-5x$ is a geometric sequence.
   Compute <u>all</u> ordered pairs (x, y) for which this is true.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2015

ROUND 5 - Trig Equations

1. ______________________

2. ______________________

3. (____________ , ____________)

1. Compute the value of $\csc(180° - A)\cos(-A)$, if $\tan A = -\frac{1}{7}$ and $\sec A < 0$.

2. Compute the degree-measures of θ over $0 \le \theta < 360°$ for which $\left(\frac{4}{81}\right)^{\sin^2\theta} + \left(\frac{4}{81}\right)^{\cos^2\theta} = \frac{12}{27}$

3. Over the interval $0° \le x < 360°$, compute the ordered pair (S, L), where S and L denote the smallest and the largest values of x respectively, that satisfy the equation

$$\left(2\cos(12x) + \sqrt{3}\right)\left(2\sin 3x - 1\right) = 0$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2015

TEAM ROUND

3 pts. 1. ________________________

3 pts. 2. ________________________

4 pts. 3. ________________________

1. Solve over the reals: $$\frac{2x}{2x-3}+\frac{9x+2x^2}{9-4x^2}=\frac{x}{2x+3}$$

2. In a 3-4-5 right triangle ABC, P is the trisection point of hypotenuse $\overline{AB}$, closer to the vertex of the larger acute angle. Compute the length of $\overline{PC}$.

3. Let $$(x,y)=\left(\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{\ddots}}}},-1+\frac{k}{1-\frac{k}{1-\frac{k}{1-\frac{k}{\ddots}}}}\right)$$
If $k=0.16$, compute the value of $x-y$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2015

Answer Sheet

Round 1

1. 20
2. 31
3. $(2,1),(5,6)$

Round 2

1. -2
2. $\left(\frac{15}{2},\frac{8}{3}\right)$
3. $(3,-2,7,-1)$

Round 3

1. 221°
2. $2n$
3. $(20,18,88,44)$

Round 4

1. 19
2. -1
3. $\left(\frac{9}{2},\frac{5}{2}\right),(-3,10)$

Round 5

1. -7
2. 45°, 135°, 225°, 315°
3. $(10,347.5)$

Team Round

1. $x \neq \pm\frac{3}{2}$ (3 pts)
2. $\frac{2}{3}\sqrt{13}$ (3 pts)
3. $\frac{9}{5}$ (4 pts)

Detailed Solutions for GBML Meet 2 - NOVEMBER 2015

ROUND 1

1. $324000 = 324\left(10^3\right) = 18^2 10^3 = 2^5 3^4 5^3$

   An odd factors will be of the form $2^0 \cdot 3^a \cdot 5^b$, where integers a and b satisfy $0 \le a \le 4$ and $0 \le b \le 3$. Thus, there are $5 \cdot 4 = \underline{20}$ factors.

2. Note that $2987 = 29 \cdot 103$ and both of these factors are prime numbers.
   Therefore, the only factors of 2987 are 1, 29, 103 and 2987.
   The only possibility is selling 103 pens at 29¢ each.
   The price reduction was **<u>31¢</u>**.

3. $A3B_{\text{base } 9} - B3A_{\text{base } 7} = 166_{\text{base } 8} \Leftrightarrow 81A + 27 + B - \left(49B + 21 + A\right) = 64 + 48 + 6 = 118_{(10)}$

   $\Leftrightarrow 80A - 48B = 112 \Leftrightarrow 5A - 3B = 7 \Leftrightarrow B = \dfrac{5A-7}{3} = A - 2 + \dfrac{2A-1}{3} \Rightarrow A = 2, 5 \Rightarrow$

   $\underline{(2,1), (5,6)}$.

   Check: $231_{(9)} - 132_{(7)} = \left(162 + 27 + 1\right) - \left(49 + 21 + 2\right) = 190 - 72 = 118_{(10)}$

   $536_{(9)} - 635_{(7)} = \left(405 + 27 + 6\right) - \left(344 + 21 + 5\right) = 438 - 370 = 118_{(10)}$

Detailed Solutions for GBML Meet 2 - NOVEMBER 2015

ROUND 2

1. $\begin{vmatrix} x & 3 \\ 4 & x-4 \end{vmatrix} = 0 \Leftrightarrow x^2 - 4x - 12 = (x-6)(x+2) = 0 \Rightarrow x = 6, -2$

   $\begin{vmatrix} -3 & x-8 \\ x & 4 \end{vmatrix} \neq 0 \Leftrightarrow -12 - x^2 + 8x \neq 0 \Leftrightarrow x^2 - 8x + 12 = (x-6)(x-2) \neq 0 \Leftrightarrow x \neq 2$ and $x \neq 6$

   Thus, the only value which satisfies both conditions is $x = \underline{-2}$

2. $\left(l + \frac{5}{2}\right)\left(w - \frac{2}{3}\right) = lw \Leftrightarrow -\frac{2}{3}l + \frac{5}{2}w = \frac{5}{3} \Rightarrow -4l + 15w = 10$

   $\left(l - \frac{5}{2}\right)\left(w + \frac{4}{3}\right) = lw \Leftrightarrow \frac{4}{3}l - \frac{5}{2}w = \frac{10}{3} \Rightarrow 8l - 15w = 20$

   Adding the equations, $4l = 30 \Rightarrow l = \frac{15}{2} \Rightarrow (l, w) = \underline{\left(\mathbf{\frac{15}{2}, \frac{8}{3}}\right)}$

3. $\begin{cases} (1) \quad A + 2B + C = 6 \\ (2) \quad 2B + C + D = 2 \\ (3) \quad C + D + A = 9 \\ (4) \quad D + A + 2B = -2 \end{cases}$ Adding the 4 equations, we have

   $3A + 6B + 3C + 3D = 15 \Rightarrow A + C + D = 5 - 2B$

   $(3) \Rightarrow 5 - 2B = 9 \Rightarrow B = -2$

   Substituting, $\begin{cases} A + C = 10 \\ C + D = 6 \\ A + D = 2 \end{cases} \Rightarrow \begin{cases} A - D = 4 \\ A + D = 2 \end{cases} \Rightarrow A = 3, D = -1, C = 7$

   Thus, $(A, B, C, D) = \underline{(\mathbf{3, -2, 7, -1})}$.

   Alternately, adding the 4 equations and dividing by 3, we have $A + 2B + C + D = 5$.
   Subtracting the first equation, $D = -1$.
   Subtracting the second equation, $A = 3$.
   Subtracting the third equation, $2B = -4 \Rightarrow B = -2$.
   Subtracting the fourth equation, $C = 7$.

Detailed Solutions for GBML Meet 2 - NOVEMBER 2015

ROUND 3

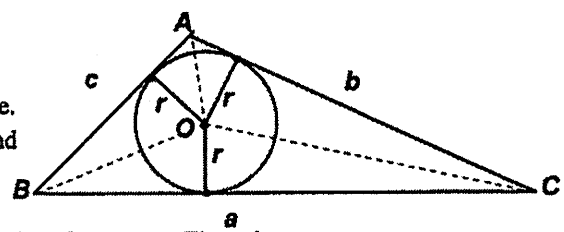
1. $22+\frac{1}{2}(180-(90-\alpha))=2(90-(180-\theta))$

   Replacing α and θ by $6x$ and $11x$ respectively, we have

   $22+45+3x=2(11x-90)\Leftrightarrow 19x=67+180=247\Rightarrow x=13$

   Thus, $m(\measuredangle\alpha)+m(\measuredangle\theta)=17x=\underline{221}°$.

2. Let K denote the area of ΔABC.
   A circle can always be inscribed in a triangle. Let O be the center of the inscribed circle and r denote its radius. A radius of the circle drawn to any of the points of tangency will be perpendicular to the side containing the point of tangency. Thus, the area of ΔABC equals the sum of the areas of triangles OAB, OBC and OCA.



   $K=\frac{1}{2}ra+\frac{1}{2}rb+\frac{1}{2}rc=r\left(\frac{a+b+c}{2}\right)$ The quantity in parentheses is half the perimeter, usually called the semi-perimeter and denoted s. Thus, for any triangle, $K=rs\Rightarrow$

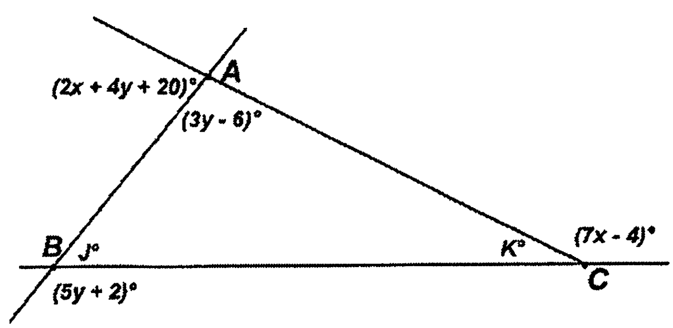
   $r=\frac{K}{s}=\frac{np}{s}=\frac{n(2s)}{s}=\underline{2n}$.

3. Supplementary angles at A: $2x+7y+14=180\Leftrightarrow 2x+7y=166$

   Triangle sum: $(3y-6)+(180-(7x-4))+(180-(5y+2))=180\Leftrightarrow 7x+2y=176$

   $7(2x+7y=166)-2(7x+2y=176)\Rightarrow 45y=7\cdot166-2\cdot176=810\Rightarrow y=18\Rightarrow x=20$

   Thus, $(x,y,J,K)=\underline{(20,18,88,44)}$.



Detailed Solutions for GBML Meet 2 - NOVEMBER 2015

ROUND 4

1. If x denote the width of the rectangle, the $(2x-1)$ denotes the length.

   $(x)(2x-1)=21 \Leftrightarrow 2x^2-x-21=(2x-7)(x+3)=0 \Rightarrow x=\frac{7}{2}$

   Thus, the rectangle is $\frac{7}{2}$ x 6 $\Rightarrow$ $P=2\left(\frac{7}{2}+6\right)=7+12=\underline{19}$.

2. The coefficient of x is the opposite of the sum of the roots. $p=-(p+q) \Leftrightarrow q=-2p$

   The constant term is the product of the roots. $pq=q \Leftrightarrow q(1-p)=0 \Leftrightarrow (q=0 \text{ or } p=1)$

   $q=0 \Rightarrow p=0$ and this case must be rejected since the roots are given to be nonzero.

   $q=1 \Rightarrow p=-2 \Rightarrow p+q=\underline{-1}$

3. The arithmetic condition requires that $y-2=\frac{(x+5)+(5-3x)}{2}=5-x \Rightarrow y=7-x$.

   The geometric condition requires that $(y+2)^2=(3-x)(9-5x) \Leftrightarrow$

   $4x^2-6x-54=0 \Leftrightarrow 2(2x-9)(x+3)=0 \Rightarrow x=\frac{9}{2},-3$

   Thus, $(x,y)=\underline{\left(\frac{9}{2},\frac{5}{2}\right),(-3,10)}$

Detailed Solutions for GBML Meet 2 - NOVEMBER 2015

ROUND 5

1. Using the standard reduction formulas, $\csc(180°-A)\cos(-A)=\csc A\cos A=\frac{\cos A}{\sin A}=\cot A$

   $\tan A=-\frac{1}{7}\Leftrightarrow\cot A=\underline{-7}$

   Note that the additional condition $\sec A<0$ restricts A to quadrant 2 (instead of 2 or 4), but this condition was not needed to evaluate the given expression.

2. $\frac{12}{27}=\frac{4}{9}=\frac{2}{9}+\frac{2}{9}=\left(\frac{4}{81}\right)^{\frac{1}{2}}+\left(\frac{4}{81}\right)^{\frac{1}{2}}$

   Thus, $\sin^2\theta=\cos^2\theta=\frac{1}{2}$ produces solutions if $\sin\theta=\cos\theta=\pm\frac{\sqrt{2}}{2}\Rightarrow$

   $\theta=\underline{\mathbf{45°,135°,225°,315°}}$.

   How do we show there are no other solutions?

   Let $x=\left(\frac{4}{81}\right)^{\sin^2\theta}$. Then: $\left(\frac{4}{81}\right)^{\cos^2\theta}=\left(\frac{4}{81}\right)^{1-\sin^2\theta}=\frac{\left(\frac{4}{81}\right)}{\left(\frac{4}{81}\right)^{\sin^2\theta}}=\frac{4}{81x}$

   Thus, $\left(\frac{4}{81}\right)^{\sin^2\theta}+\left(\frac{4}{81}\right)^{\cos^2\theta}=\frac{12}{27}\Leftrightarrow x+\frac{4}{81x}=\frac{4}{9}\Rightarrow 81x^2-36x+4=(9x-2)^2=0\Rightarrow x=\frac{2}{9}$

   and the above results are the only solutions over $[0,360°)$.

3. $\left(2\cos(12x)+\sqrt{3}\right)(2\sin3x-1)=0\Leftrightarrow\begin{cases}2\cos(12x)+\sqrt{3}=0\\2\sin3x-1=0\end{cases}\Leftrightarrow\begin{cases}\cos(12x)=-\frac{\sqrt{3}}{2}\\\sin3x=\frac{1}{2}\end{cases}$

   $\Rightarrow 12x\in Q2,3$ or $3x\in Q1,2$

   $12x=\begin{cases}150°+n\cdot360°\\210°+n\cdot360°\end{cases}\Leftrightarrow x=\begin{cases}12.5°+n\cdot30°\\17.5°+n\cdot30°\end{cases}\Rightarrow x_{min}=12.5°,\ x_{max}=347.5°$

   $3x=\begin{cases}30°+n\cdot360°\\150°+n\cdot360°\end{cases}\Leftrightarrow x=\begin{cases}10°+n\cdot120°\\50°+n\cdot120°\end{cases}\Rightarrow x_{min}=10°,\ x_{max}=290°$

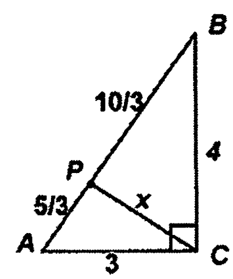
   Thus, $(S,L)=\underline{(10,347.5)}$.

Detailed Solutions for GBML Meet 2 - NOVEMBER 2015

TEAM ROUND

1. Clearly, to avoid division by 0, $x \neq \pm\frac{3}{2}$. Transposing terms, $\frac{2x}{2x-3}+\frac{9x+2x^2}{9-4x^2}=\frac{x}{2x+3} \Leftrightarrow$
$\frac{9x+2x^2}{9-4x^2}=\frac{x}{2x+3}-\frac{2x}{2x-3}$ and the solutions will be equivalent. Combining terms on the right-hand side,
$$\frac{x}{2x+3}-\frac{2x}{2x-3}=\frac{x(2x-3)-2x(2x+3)}{(2x+3)(2x-3)}=\frac{2x^2-3x-4x^2-6x}{(2x+3)(2x-3)}=\frac{-(9x+2x^2)}{4x^2-9}=\frac{9x+2x^2}{9-4x^2}$$
Thus, this equation is an identity and is true for $\boxed{\text{all reals } \underline{\text{except}}\ x=\pm\frac{3}{2}}$ or $\underline{x\neq\pm\frac{3}{2}}$

2. Using Stewart's formula, we have
$$16\cdot\frac{5}{3}+9\cdot\frac{10}{3}=5x^2+5\cdot\frac{5}{3}\cdot\frac{10}{3}\Leftrightarrow 5x^2=\frac{510-250}{9}\Leftrightarrow x^2=\frac{52}{9}\Rightarrow x=\underline{\frac{2}{3}\sqrt{13}}$$



3.
$$x=\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{\ddots}}}}\Leftrightarrow x=\frac{1}{2-x}\Rightarrow 2x-x^2=1\Leftrightarrow x^2-2x+1=(x-1)^2=0\Rightarrow x=1$$

$$y+1=\boxed{\cfrac{k}{1-\boxed{\cfrac{k}{1-\cfrac{k}{1-\cfrac{k}{\ddots}}}}}}\Leftrightarrow y+1=\frac{k}{1-(y+1)}\Rightarrow -y^2-y=k\Leftrightarrow y^2+y+k=0$$

Substituting for k, $25y^2+25y+4=(5y+1)(5y+4)=0\Rightarrow y=-\frac{1}{5},-\frac{4}{5}$ Which is correct?

Examining the sequence of truncated continued fractions,

$$-1+\boxed{0.16}=-0.84,\ -1+\frac{0.16}{0.84}\approx-1+\boxed{0.19}\approx-0.81,\ -1+\frac{0.16}{\approx 0.81}\approx-1+\boxed{0.198}\approx-0.802$$

Notice the $\boxed{\text{adjustment term}}$ is getting closer and closer to $\frac{1}{5}$ and we have $y=-\frac{4}{5}$

Thus, $x-y=\underline{\frac{9}{5}}$.