GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2016

ROUND 1 - Arithmetic: Open

1. ______________________

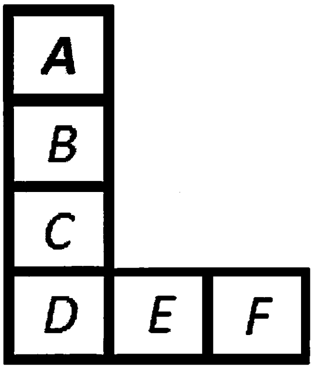
2. ______________________

3. ______________________

1. N is odd, positive, divisible by 3 and has exactly 6 divisors.
   Determine the <u>smallest</u> possible value of N.

2. Exactly 4 integers between 575 and 600 are prime.
   The 4-digit sum of these primes is $abc6$ and $a + b + c = 10$.
   Compute the product $a \cdot b \cdot c$.

3. How many distinct ways can you arrange these numbers 1, 3, 6, 7, 8, and 11 into the positions A, B, C, D, E, and F so that the sum of the numbers in the vertical column equals the sum of the numbers in the horizontal row.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2016

ROUND 2 - Simultaneous Linear equations, Word Problems, Matrices

1. ______________________________

2. ______________________________

3. ______________________________

1. Given: $a+2b+3c=42$ and $a:b:c=1:2:3$.
   Compute $a+b+c$.

2. In a river with a steady current, it takes Wonder Woman 8 minutes to swim a distance upstream, but it takes her only 4 minutes to swim back. How many <u>minutes</u> would it take a Wonder Woman doll to float the same distance downstream?

3. Compute the value of the following determinant: $\begin{vmatrix} 91 & \frac{3}{14} & 1 \\ 42 & \frac{9}{42} & 3 \\ 119 & \frac{4}{21} & 2 \end{vmatrix}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2016

ROUND 3 - Geometry: Angles and Triangles

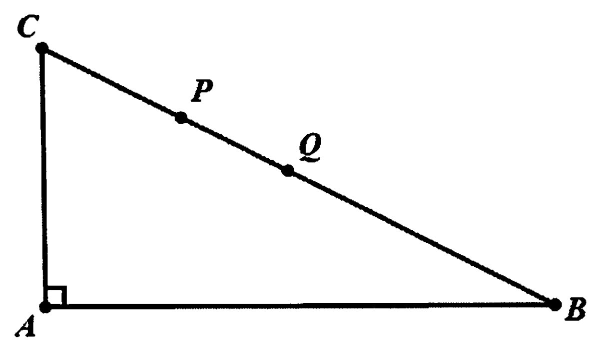
1. ______________________________

2. ______________________________

3. ______________________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. In right triangle CAB, $AC = CQ = x$, $AB = BP = y$, and $PQ = 1$. Find the value y in terms of x.



2. The lengths of the sides of a non-isosceles triangle, given in increasing order of size, are 8, x, and 14. Determine <u>all</u> possible integer values of x.

3. The lengths of the sides of a triangle are 25, 29, and 36. There is a point P on the longest side of the triangle whose distance from the opposite vertex is 20. Compute the distance from the point P to the midpoint of the shortest side.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2016

ROUND 4 - Algebra 2: Quadratic Equations, Problems involving them and Theory of Quadratics

1. ______________________

2. (__________ , __________)

3. ______________________

1. Compute the value of the constant K for which the product of the roots of the equation $3x^2 - K = Kx - 1$ when multiplied by the reciprocal of the sum of the roots equals $-\frac{3}{4}$ times the sum of the roots.

2. Compute the ordered pair of numbers (x, y) for which $(k-3)$ is a factor of both $k^2 - (x+y)k + 3y$ and $(x-1)k^2 + yk + x$.

3. Compute all ordered pairs of real numbers (x, y) for which $17x^2 + 2y^2 + 2 = 10xy + 6x$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2016

ROUND 5 - Trig Equations

1. ______________________

2. ______________________

3. ______________________

1. Determine <u>all</u> values of x in set $S = \left\{ x \mid 3\tan^2 x + 4\sec x + 4 = 0 \text{ and } 0^\circ \le x \le 360^\circ \right\}$.
   Note: x must be expressed in degrees.

2. Determine, in degrees, <u>all</u> values of x for which $\left(\frac{2}{5}\right)^{\sin x} < 1$, given $90^\circ \le x \le 270^\circ$.

3. Compute <u>all</u> possible values of $\sec x$, if $\sin^4 x - 2\sin x\cos x + 2\sin^2 x\cos^2 x + \cos^4 x = 0$ and $0 \le x < 36^\circ$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2016

TEAM ROUND

3 pts. 1. ______________________________

3 pts. 2. ______________________________

4 pts. 3. ______________________________

1. In triangle ABC, the measure of the <u>exterior</u> angle at C is equal to four times the measure of the angle at A. An <u>exterior</u> angle at A has a measure which is 19° more than twice the measure of the angle at B. Compute the degree measure of the angle at A.

2. Given $5x-3y-5z=4x-15y+10=0$, where x, y and z are <u>positive</u> integers.
   For a <u>minimum</u> value of x, compute the ratio of $x:y:z$.

3. A single stamp is selected from one of two sheets.
   Sheet #1 has n rows of stamps, each containing n stamps.
   Sheet #2 has $(n + 1)$ rows of stamps, each containing $(n + 1)$ stamps.
   The stamp may not be a border stamp (neither top or bottom row not left or right column) and it may not be located on a diagonal (upper left to lower right or vice versa).
   This stamp may be chosen in $A(n-B)(n-C)$ ways, where A, B, and C are integersand $B<C$. Determine the ordered triple (A,B,C).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2016

Answer Sheet

Round 1

1. 45
2. 30
3. 24

Round 2

1. 18
2. 16
3. 28

Round 3

1. $\dfrac{2x-1}{2(x-1)}$
2. 9, 10, 11, 12, 13
3. $\dfrac{25}{2}$ (or 12.5)

Round 4

1. 2
2. $(3,-7)$
3. $\left(\dfrac{2}{3},\dfrac{5}{3}\right)$

Round 5

1. 180 (or 180°)
2. $90° \le x < 180°$ or $[90°,180°)$ (degree symbols not required)
3. $\pm\sqrt{2}$

Team Round

1. 23 (3 pts)
2. 35: 10 : 29 (3 pts)
3. $(2,2,3)$ (4 pts)

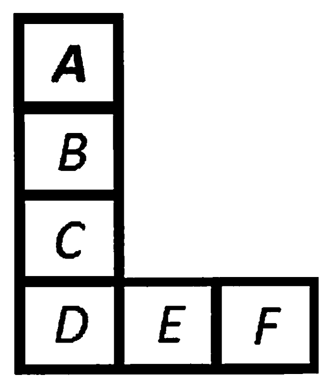
Detailed Solutions for GBML Meet 2 - NOVEMBER 2016

ROUND 1

1. Here's a listing:
   3: 1, 3
   9: 1,3,9
   15: 1, 3, 5, 15
   21: 1, 3, 7, 21
   27: 1, 3, 9, 27
   33: 1, 3, 11, 33
   39: 1, 3, 13, 39
   <u>45</u>: 1, 3, 5, 9, 15, 45 Bingo!

2. Eliminating even integers, multiples of 3 and multiples of 5, we must check:
   577, 581, 583, 587, 589, 593 and 599
   583 is divisible by 11
   581 is divisible by 7
   This leaves 577, 587, 589, 593 and 599
   Since the required sum ends in 6, either 589 or 599 is not prime.
   The corresponding sums $(577+587+593)+\begin{cases}589\\599\end{cases}=175\underline{7}+\begin{cases}589\\599\end{cases}$ are either 2346 or 2356.
   Only the latter satisfies the digit-sum requirement. Thus, $a \cdot b \cdot c = \underline{\mathbf{30}}$.

3. We require that $A+B+C=E+F$.
   Since the sum of the given 6 numbers is 36, we must exclude one number and subdivide the remaining five into groups of 3 numbers and 2 numbers which have the <u>same sum</u>. This is only possible for 8 $(\{3,11\},\{1,6,7\})$ and for 6 $(\{7,8\},\{1,3,11\})$.
   In each case, we must pick *D*, arrange the horizontal numbers {*E*, *F*}, and then arrange the vertical numbers {*A*, *B*, *C*}.
   Therefore, the total number of arrangements is $2(1\cdot 2!\cdot 3!)=\underline{\mathbf{24}}$.



Detailed Solutions for GBML Meet 2 - NOVEMBER 2016

ROUND 2

1. Let $(a,b,c)=(n,2n,3n)$. Then: $n+2(2n)+3(3n)=14n=42 \Rightarrow n=3 \Rightarrow (a,b,c)=(3,6,9)$
   Therefore, $a+b+c=\underline{\mathbf{18}}$.

2. Assume the current (speed of the water flow downstream) is c miles/minute and that Wonder Woman swam at w miles/minute in still water.
   Assume also that the distance she swam (upstream and downstream) was d miles.
   Her upstream and downstream rates were $w-c$ and $w+c$ respectively.
   $RT=D \Rightarrow (w-c)8=(w+c)4=d \Rightarrow 12c=4w \Rightarrow w=3c$
   Substituting, $d=16c$
   The doll will float downstream with at the speed of the current for t minutes.
   $ct=16c \Rightarrow t=\underline{\mathbf{16}}$ minutes.

3. Note the values in (row, column) positions (1,2) and (2,2) are equivalent $\frac{9}{42}=\frac{3}{14}$.

   Strategy:
   Replace row 1 with row 1 − row 2.
   Replace row 3 with row 3 + row 1.
   Using minors, expand the last determinant by column 2.
   Using $\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$, evaluate each 2 x 2 determinant.

$$\begin{vmatrix} 91 & \frac{3}{14} & 1 \\ 42 & \frac{9}{42} & 3 \\ 119 & \frac{4}{21} & 2 \end{vmatrix}$$

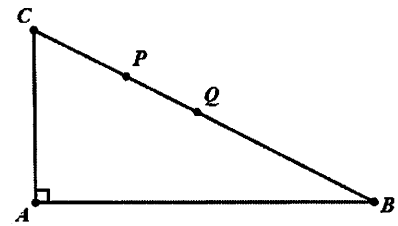
$$\begin{vmatrix} 91 & \frac{3}{14} & 1 \\ 42 & \frac{9}{42} & 3 \\ 119 & \frac{4}{21} & 2 \end{vmatrix} = \begin{vmatrix} 49 & 0 & -2 \\ 42 & \frac{3}{14} & 3 \\ 119 & \frac{4}{21} & 2 \end{vmatrix} = \begin{vmatrix} 49 & 0 & -2 \\ 42 & \frac{3}{14} & 3 \\ 168 & \frac{4}{21} & 0 \end{vmatrix} = 49\begin{vmatrix} \frac{3}{4} & 3 \\ \frac{4}{21} & 0 \end{vmatrix} - 0\begin{vmatrix} 42 & 3 \\ 168 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 42 & \frac{3}{14} \\ 168 & \frac{4}{21} \end{vmatrix}$$

$$49\left(-\frac{4}{21}\cdot 3\right)-2\left(42\cdot\frac{4}{21}-168\cdot\frac{3}{14}\right)=-28-2(8-12\cdot 3)=-28+2\cdot 28=\underline{\mathbf{28}}$$

Detailed Solutions for GBML Meet 2 - NOVEMBER 2016

ROUND 3

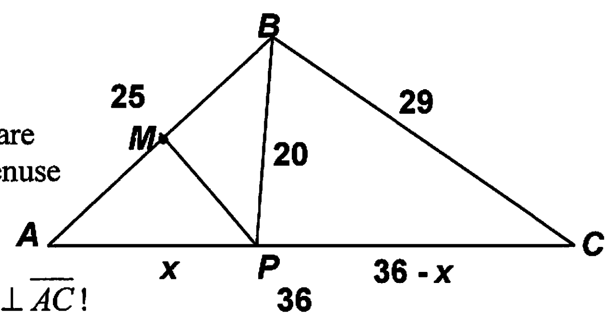
1. $AC = CQ = x$, $AB = BP = y$, and $PQ = 1 \Rightarrow \begin{cases} CP = x-1 \\ BQ = y-1 \end{cases}$



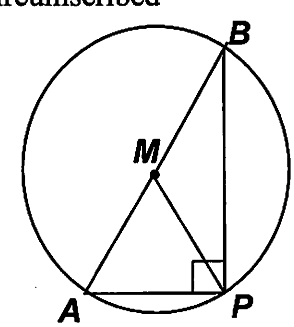
   By the Pythagorean theorem,
   $\cancel{x^2} + \cancel{y^2} = (x+y-1)^2 = \cancel{x^2} + \cancel{y^2} + 1 + 2xy - 2x - 2y$.
   Thus, $2xy - 2y = 2x - 1 \Leftrightarrow 2y(x-1) = 2x-1 \Leftrightarrow y = \underline{\mathbf{\dfrac{2x-1}{2(x-1)}}}$

2. Since the sides are in increasing order and the triangle is not isosceles, $x \geq 9$ and $x \leq 13$.
   For each of these integer values, the Triangle Inequality is satisfied.
   Thus, $x = $ **<u>9, 10, 11, 12, 13</u>**.

3. Solution #1: ($\overline{BP} \perp \overline{AC}$?)



   Recalling common Pythagorean Triples, 29 and 25 are interesting possibilities for the lengths of the hypotenuse in a right triangle (20-21-29 and 15-20-25).
   $x = 15 \Rightarrow 36 - x = 21$ and we $\overline{BP}$ has divided ΔABC into two right triangles, confirming that $\overline{BP} \perp \overline{AC}$!
   In ANY right triangle, the midpoint of the hypotenuse is the center of the circumscribed circle, since an angle inscribed in a semicircle must be a right angle!!



   Therefore, $MA = MB = MP = \underline{\mathbf{\dfrac{25}{2}}}$.

   Solution #2: (Ouch - Applying Stewart's Theorem to ΔABC)
   $25^2(36-x) + 29^2 x = 20^2 \cdot 36 + 36x(36-x)$
   $\Rightarrow 22500 - 625x + 841x = 14400 + 1296x - 36x^2$
   $\Rightarrow 36x^2 - 1080x + 8100 = 0$
   $\Rightarrow x^2 - 30x + 225 = (x-15)^2 = 0 \Rightarrow x = 15$ and the same result follows.

Detailed Solutions for GBML Meet 2 - NOVEMBER 2016

ROUND 4

1. $3x^2 - K = Kx - 1 \Leftrightarrow 3x^2 - Kx + (1-K) = 0$

   The product of the roots is $\frac{1-K}{3}$. The sum of the roots is $\frac{K}{3}$.

   Thus, $\frac{1-K}{3} \cdot \frac{3}{K} = -\frac{3}{4} \cdot \frac{K}{3} \Rightarrow \frac{1-K}{k} = \frac{-K}{4} \Rightarrow 4 - 4K = -K^2$

   $\Leftrightarrow K^2 - 4K + 4 = (K-2)^2 = 0 \Rightarrow K = \underline{\mathbf{2}}$.

2. Treat $k^2 - (x+y)k + 3y$ as a trinomial in k with coefficients 1, $-(x+y)$, and $3y$

   Evaluate for $k = 3$, using synthetic substitution.

   The remainder must be zero, if $(k-3)$ is to be a factor.

	1	$-(x+y)$	$3y$
$3\|$	1	$-x-y+3$	$3y-3x-3y+9 = 9-3x = 0 \Rightarrow x = 3$

   Similarly, for the second trinomial

	$x-1$	y	x
$3\|$	$x-1$	$y+3x-3$	$x+3y+9x-9 = 10x+3y-9$

   Substituting $x = 3$ and setting equal to 0, we have $30 + 3y - 9 = 0 \Rightarrow y = -7$.

   Therefore, $(x, y) = \underline{\mathbf{(3,-7)}}$.

3. $17x^2 + 2y^2 + 2 = 10xy + 6x \Leftrightarrow 2y^2 + (-10x)y + (17x^2 - 6x + 2) = 0$

   This equation is a quadratic in terms of y.

   Applying the quadratic formula, $y = \frac{10x \pm \sqrt{100x^2 - 8(17x^2 - 6x + 2)}}{4}$.

   The radicand simplifies to

   $100x^2 - 8(17x^2 - 6x + 2) = -36x^2 + 48x - 16 = -4(9x^2 - 12x + 4) = -4(3x-2)^2$

   The only way the radicand can be non-negative and the roots real if $x = \frac{2}{3} \Rightarrow (x, y) = \underline{\left(\mathbf{\frac{2}{3}, \frac{5}{3}}\right)}$.

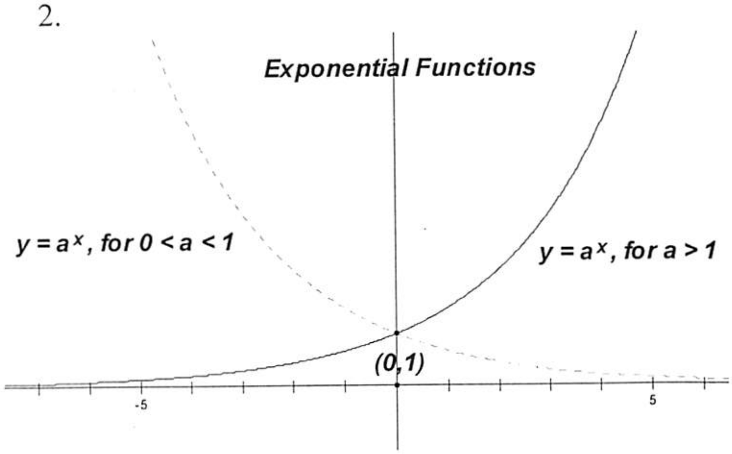
Detailed Solutions for GBML Meet 2 - NOVEMBER 2016

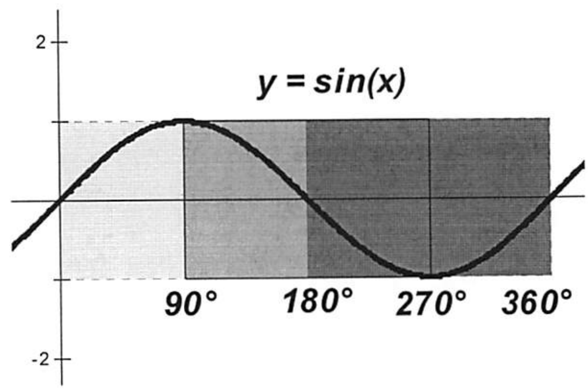
ROUND 5

1. $3\tan^2 x + 4\sec x + 4 = 0 \Leftrightarrow 3\left(\sec^2 x - 1\right) + 4\sec x + 4 = 3\sec^2 + 4\sec x + 1 = 0$

   $\left(3\sec x + 1\right)\left(\sec x + 1\right) = 0 \Rightarrow \sec x = -\frac{1}{3}, -1 \Rightarrow \cos x = \cancel{-3}, -1 \Rightarrow x = \underline{\mathbf{180}}°$

2.





The given exponential expression behaves like the dotted graph on the left above.
It decreases in value from left to right and passes through (0, 1).
Specifically, when the exponent is negative, the functional value is greater than 1; when the exponent is positive the functional value is less than 1.
From the graph at the right, we see that $y = \sin x$ is a strictly decreasing function over the closed interval $[90°, 270°]$; decreasing through positive values over the interval $[90°, 180°]$ and decreasing through negative values over $[180°, 270°]$

At $x = 90°$, we have $\left(\frac{2}{5}\right)^1 < 1$ (check) and at $x = 180°$ we have $\left(\frac{2}{5}\right)^0 = 1$ (rejected).

Thus, the required interval is $\underline{\mathbf{90° \le x < 180°}}$ or $\underline{\mathbf{[90°, 180°)}}$.

3. $\sin^4 x - 2\sin x\cos x + 2\sin^2 x\cos^2 x + \cos^4 x = 0$

   $\Leftrightarrow \left(\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x\right) - 2\sin x\cos x = 0$

   $\Leftrightarrow \left(\sin^2 x + \cos^2 x\right)^2 - \sin 2x = 0$

   $\Leftrightarrow 1 - \sin 2x = 0$

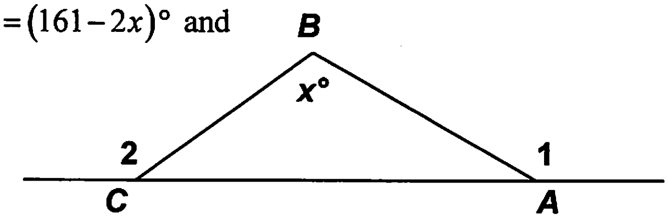
   $\Rightarrow 2x = 90° + n\left(360°\right) \Rightarrow x = 45° + n\left(180°\right)$ - quadrants 1 and 3.

   Therefore, $\sec x = \underline{\pm\sqrt{\mathbf{2}}}$.

Detailed Solutions for GBML Meet 2 - NOVEMBER 2016

TEAM ROUND

1. If $m\angle B = x°$, then $m\angle 1 = (2x+19)° \Rightarrow m\angle BAC = (161-2x)°$ and $m\angle 2 = 4(161-2x)°$.



Since the measure of any exterior angle of a triangle equals the sum of the measures of the two remote interior angles, we have

$m\angle 2 = m\angle ABC + m\angle BAC \Leftrightarrow 4(161-2x) = x + (161-2x)$

$\Leftrightarrow 4\cdot 161 - 8x = 161 - x$

$\Leftrightarrow 3\cdot 161 = 7x$

$\Leftrightarrow x = 3\cdot 23 = 69 \Rightarrow m\angle BAC = 161 - 2\cdot 69 = \underline{\mathbf{23}}$.

2. $5x - 3y - 5z = 0 \Rightarrow z = x - \frac{3}{5}y$

Subtracting, $\begin{cases} 5x-3y-5z=0 \\ 4x-15y+10=0 \end{cases} \Rightarrow x + 12y - 5z = 10$

Substituting for z, $x + 12y - 5\left(x - \frac{3}{5}y\right) = 10 \Rightarrow -4x + 15y = 10 \Rightarrow y = \frac{4x+10}{15}$

Therefore, $z = x - \frac{3}{5}y = x - \frac{3}{5}\left(\frac{4x+10}{15}\right) = x - \frac{4x+10}{25} = \frac{21x-10}{25}$ and

$x:y:z = x:\left(\frac{4x+10}{15}\right):\left(\frac{21x-10}{25}\right) = 75x:(20x+50):(63x-30)$

For $x = 5, 20, 35, \ldots$, y is an integer, but we also require that $(21x-10)$ be a multiple of 25 and this is the case first for $x = 35$.

$\frac{21\cdot 5-10}{25} = \frac{95}{25}$, $\frac{21\cdot 20-10}{25} = \frac{410}{25}$, $\frac{21\cdot 35-10}{25} = \frac{725}{25} = 29 \Rightarrow x:y:z = \underline{\mathbf{35:10:29}}$.

TEAM ROUND - continued

3. Consider some specific cases:
   $n \le 4$: none $n = 5 : 4$ $n = 6 : 8$ $n = 7 : 16$
   A clear pattern emerges – the total *doubles* each time.
   For a specific *n*, there are $2^{(n-3)}$ choices.
   For our 2 sheets, there are $2^{(n-3)} + 2^{(n-2)} = 2^{(n-3)}(1+2) = 3 \cdot 2^{(n-3)}$ choices.
   Unfortunately, this is not in the required form.
   From another perspective, we count the border stamps and the diagonal stamps and subtract this sum from the total for each sheet.
   Border stamps: $4n-4$ (for even or odd values of *n*)
   Diagonal stamps: For even *n*, $2(n-2)$. For odd *n*, $2(n-2)-1$
   Thus, the total number of choices is $\begin{cases} n^2-4n+4-2(n-2) = n^2-6n+8, & \text{for even } n \\ n^2-6n+9, & \text{for odd } n \end{cases}$
   Since consecutive integers have opposite parity, we are adding one of each.
   For *n* even, $T = (n^2-6n+8) + ((n+1)^2 - 6(n+1) + 9) = 2n^2 - 10n + 12 = 2(n-2)(n-3)$.
   For *n* odd, $T = (n^2-6n+9) + ((n+1)^2 - 6(n+1) + 8)$, producing the same result.
   Therefore, $(A,B,C) = \underline{(\mathbf{2,2,3})}$.