GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2017

ROUND 1 - Arithmetic: Open

1. ______________________________

2. (____________ , ____________)

3. ______________________________

1. An automobile odometer reads 15951. When the automobile is driven k more miles, the odometer reading will again be a palindrome (a number which reads the same from left to right and from right to left). Compute the minimum possible nonzero value of k.

2. If $A+B<24$, compute the ordered pair of positive integers (A,B) for which $134_A + 25_B = 76_{(A+B)}$.

3. A six-digit number $N = ABCDEF$ contains the digits 1, 2, 3, 4, 5, and 6. Determine all possible values N for which the three-digit number ABC is divisible by 4, the three-digit number BCD is divisible by 5, the three-digit number CDE is divisible by 3, and the three-digit number DEF is divisible by 11.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2017

ROUND 2 - Simultaneous Linear equations, Word Problems, Matrices

1. Paul: __________ Cory: __________

2. (________ , ________)

3. { ____________________________ }

1. Paul and Cory working together can complete a job in 6 days. After they have been working together for 4 days, Paul leaves and Cory finishes the job alone in 8 more days. Compute the number of days each would require to do the entire job alone.

2. Compute the ordered pair (p,q) for which $\begin{cases} \dfrac{4p+q}{3}-\dfrac{p-2q}{6}=\dfrac{p-4}{6} \\ \dfrac{1}{3}+\dfrac{p+1}{2}-\dfrac{q-2}{4}=\dfrac{p}{12} \end{cases}$.

3. Compute $S=\left\{(a,b)\ \middle|\ \dfrac{\dfrac{2a+b}{a+b}-1}{1-\dfrac{b}{a+b}}=a-\dfrac{b}{2} \text{ and } \dfrac{b}{8}=\dfrac{1}{4+\dfrac{2a}{1-\dfrac{a}{2}}}\right\}$

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 3 - Geometry: Angles and Triangles

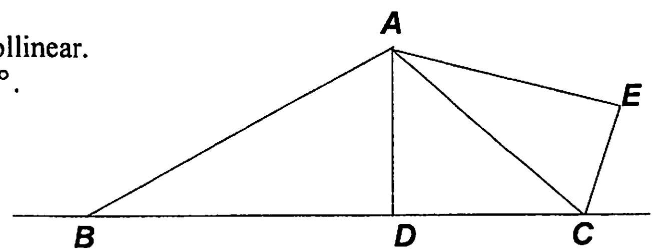
1. ______________________

2. (______ , ______ , ______)

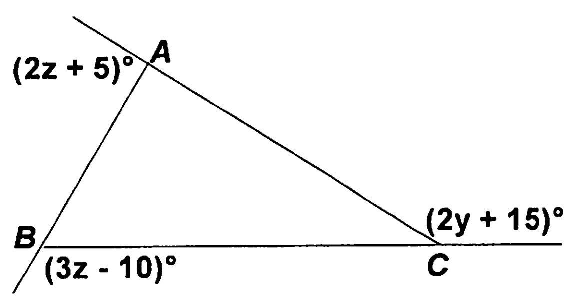
3. __________ : __________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

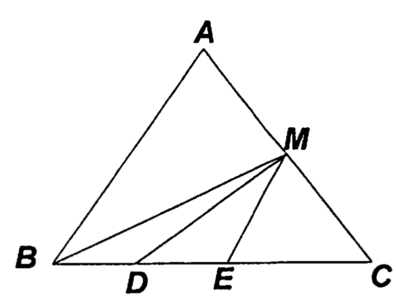
1. In the diagram at the right, *B*, *D*, and *C* are collinear. $m\angle ABC = 30°, m\angle ACD = 45°, m\angle CAE = 30°$. $AE = 6\sqrt{2}$, $\overline{AD} \perp \overline{BC}$, $\overline{AE} \perp \overline{CE}$ Compute *BD*.



2. In ΔABC, $m\angle BAC = (x+30)°$, $m\angle ACB = (x+20)°$, $m\angle ABC = (y-10)°$, and the exterior angles have the measures indicated in the diagram at the right. Compute the ordered triple (x, y, z).



3. ΔABC is equilateral and *M* is the midpoint of $\overline{AC}$. $BE : EC = 7:5$, $BD : DC = 1:3$. Compute the ratio of the area of ΔMDE to the area of ΔABM.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2017

ROUND 4 - Algebra 2: Quadratic Equations, Problems involving them and Theory of Quadratics

1. ______________________

2. ______________________

3. $k =$ ______________________

1. Given: $\dfrac{\frac{y-1}{2}}{1+\frac{2y}{3}} = \dfrac{2}{5y}\left(\dfrac{3y}{2} - 2\right)$

   Compute the <u>positive</u> <u>difference</u> between the sum of the roots of this equation and the product of the roots of this equation.

2. Compute <u>all</u> values of k for which the roots of $kx^2 + 4k - 1 = 6kx - 2x^2$ will <u>not</u> be real.

3. For positive numbers k and j, the roots of the equation $x^2 + kx + j = 0$ differ by 1. Determine a simplified expression for k in terms of j.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2017

ROUND 5 - Trig Equations

1. ______________________

2. ______________________

3. ______________________

1. Compute all values of $\sin\theta$ for which $24\tan\theta = 5\cos\theta$ and $0 < \theta < \pi$.

2. If $\cos x + \sin x = p$ and $\cos 2x = q$, determine a simplified expression for q in terms of p, given that $p < 0$ and $q > 0$.

3. Given: $\tan A = -\frac{1}{\sqrt{2}}$, $\sec A < 0$

   Compute all values of x over $0° \le x < 360°$ for which

   $$\frac{(\cos 495°)\cot A + \csc A(\tan 210°)}{\tan A \cdot \sec A \cdot \csc x} = \sec 300°.$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2017

TEAM ROUND

3 pts.	1.	____________________
3 pts.	2.	(________ , ________ , ________)
4 pts.	3.	____________________

1. In rectangle $ABCD$, $AD = 10$, $CD = 15$, P is a point in the interior of the rectangle such that $PB = 9$ and $PA = 12$. Compute PD.

2. The roots of the equation $x^2 + 3x + k = 0$ are p and q, where k is some positive integer constant. The quadratic equation $Ax^2 + Bx + C = 0$, where A, B, and C are integers and $A > 0$, has roots $\dfrac{3}{2p-1}$ and $\dfrac{3}{2q-1}$. Compute the ordered triple (A, B, C), where A is given as a simplified expression in terms of k.

3. Compute <u>all</u> values of x over $0 \le x < 360°$ for which

$$\begin{cases} \sin x \cdot \sin 3x \cdot \cos 3x = \left(\sin^2 150°\right) \cdot \sin x \\ \sin x \ge 0 \end{cases}$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2017

Answer Sheet

Round 1

1. 110
2. $(8,7)$
3. 324561

Round 2

1. $(8, 24)$
2. $(-2, 2)$
3. $\left(\frac{4}{3}, \frac{2}{3}\right)$

Round 3

1. 12
2. $(45,50,50)$
3. 1 : 3

Round 4

1. $\frac{7}{3}$
2. $\frac{2}{5} < k < 1$
3. $k = \sqrt{1+4j}$

Round 5

1. $\frac{1}{5}$
2. $q = -p\sqrt{2-p^2}$
3. 60°, 120°

Team Round

1. 10 (3 pts)
2. $(4k+7,24,9)$ (3 pts)
3. 0, 5, 25, 65, 85 (4 pts)
   125, 145, 180
   (in any order)

Detailed Solutions for GBML Meet 2 - NOVEMBER 2017

ROUND 1

1. To get the smallest difference, we want to change the least significant digits by a minimum amount. However, changing the units digit would require that we also change the leftmost digit. So, we add 1 to the tens digit (and thousands digit) and roll the hundreds digit to zero. Thus, the minimum difference is $16\underline{0}61 - 15951 =$ **<u>110</u>**.

2. $134_A + 25_B = 76_{(A+B)} \Rightarrow A$ must be at least 5 and B must be at least 6.

$\Leftrightarrow (1A^2 + 3A + 4) + (2B + 5) = 7(A + B) + 6$

$\Leftrightarrow A^2 - 4A + 4 = 5B + 1$

$\Leftrightarrow (A-2)^2 = 5B + 1$

$\Leftrightarrow A - 2 = \sqrt{5B+1}$

Thus, B must be selected so that $5B+1$ is a perfect square.

$B = \cancel{3}, 7, 16, \ldots \Rightarrow A = 2+6 = 8,\ 2+9 = 11, \ldots$

Thus, $(A,B) = \underline{(\mathbf{8,7})}$ is unique, since $11 + 16 > 24$.

3. Since BCD must be divisible, $D = 5$. (0 is not available.)
Since ABC must be divisible by 4, we have
$(B,C) = (1,2),(1,6),(2,4),(3,2),(3,6),\cancel{(5,2)},\cancel{(5,6)},(6,4)$
Since DEF must be divisible by 11, $D+F-E$ must be a multiple of 11.
The maximum value of $D+F-E$ is 10 (if $F = 6$ and $E = 1$. So, the only possible multiple of 11 is 0 and $F - E = -5 \Rightarrow (E,F) = (6,1)$.
Thus, $(B,C) = (2,4),(3,2) \Rightarrow ABCDEF = 324561$ or 432561
But we also require that CDE be divisible by 3.
Only the first arrangement satisfies this condition and **<u>324561</u>** is unique.

Detailed Solutions for GBML Meet 2 - NOVEMBER 2017

ROUND 2

1. In 4 days, $\frac{2}{3}$ of the job has been completed, leaving $\frac{1}{3}$ still to be done. Assuming Cory and Paul can each do the job alone in c and p hours respectively, we have $8\left(\frac{1}{c}\right)=\frac{1}{3} \Rightarrow c=\underline{\mathbf{24}}$ and $6\left(\frac{1}{p}+\frac{1}{24}\right)=1 \Rightarrow \frac{6}{p}=\frac{3}{4} \Rightarrow p=\underline{\mathbf{8}}$.

2. $\begin{cases} \dfrac{4p+q}{3}-\dfrac{p-2q}{6}=\dfrac{p-4}{6} \\ \dfrac{1}{3}+\dfrac{p+1}{2}-\dfrac{q-2}{4}=\dfrac{p}{12} \end{cases}$

Multiplying both sides of the first equation by 6, and the second by 12, we have

$$\begin{cases} 8p+2q-p+2q=p-4 \\ 4+6p+6-3q+6=p \end{cases} \Leftrightarrow \begin{cases} 6p+4q=-4 \\ 5p-3q=-16 \end{cases} \Leftrightarrow \begin{cases} 18p+12q=-12 \\ 20p-12q=-64 \end{cases} \Rightarrow 38p=-76 \Rightarrow p=-2$$

Substituting back in the $3p+2q=-2$, we have $(p,q)=\underline{(\mathbf{-2,2})}$.

3. $$\frac{\frac{2a+b}{a+b}-1}{1-\frac{b}{a+b}}=a-\frac{b}{2} \Rightarrow \frac{2a+b-(a+b)}{a+b-b}=\frac{2a-b}{2} \Rightarrow \frac{a}{a}=1=\frac{2a-b}{2} \Rightarrow b=2a-2$$

$$\frac{b}{8}=\frac{1}{4+\frac{2a}{1-\frac{a}{2}}} \Rightarrow \frac{b}{8}=\frac{1}{4+\frac{4a}{2-a}}=\frac{2-a}{8} \Rightarrow b=2-a$$

Therefore, $2a-2=2-a \Rightarrow 3a=4 \Rightarrow S=\underline{\left\{\left(\mathbf{\frac{4}{3},\frac{2}{3}}\right)\right\}}$.

Detailed Solutions for GBML Meet 2 - NOVEMBER 2017

ROUND 3

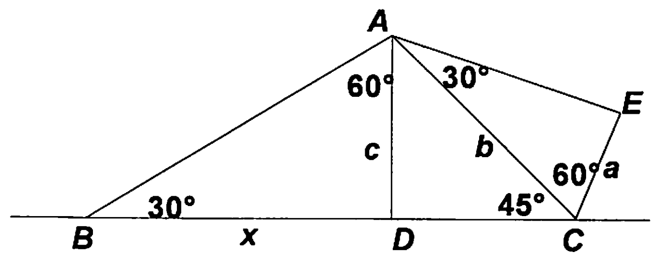
1.
$AE = 6\sqrt{2}$

$\Rightarrow a = \left(\dfrac{6\sqrt{2}}{\sqrt{3}}\right) = 2\sqrt{6}$

$\Rightarrow b = 2\left(2\sqrt{6}\right) = 4\sqrt{6}$

$\Rightarrow c = \dfrac{4\sqrt{6}}{\sqrt{2}} = 4\sqrt{3}$

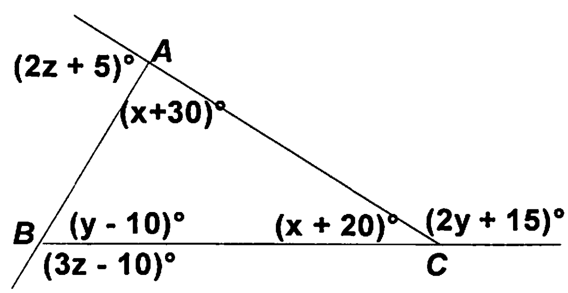
Thus, $x = \left(4\sqrt{3}\right)\cdot\sqrt{3} = \underline{\mathbf{12}}$.



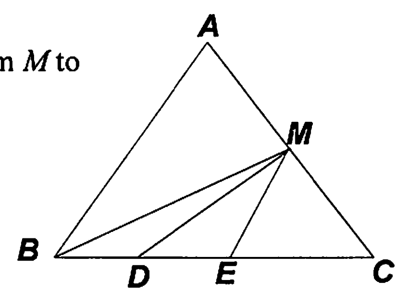
2. Interior angles: $2x + y + 40 = 180$
Exterior angles: $2y + 5z + 10 = 360$

Supplementary pairs: $\begin{cases} \boxed{x + 2z = 145} \\ y + 3z = 200 \\ \boxed{x + 2y = 145} \end{cases}$.

Subtracting the boxed equations, we have $y = z$. Substituting in the 2nd supplementary equation, we have $y = z = 50$. Substituting in the interior angles equation, we have $(x, y, z) = \underline{\mathbf{(45,50,50)}}$.



3. Since M is the midpoint of $\overline{AC}$, the area of ΔMBA equals the area of ΔMBC. Since triangles MDE and MBC share a common altitude from M to $\overline{BC}$, their areas are in the ratio of DE to BC.
$BE : EC = 7 : 5 \Rightarrow BE = 7a,\ EC = 5a,\ BC = 12a$
$BD : DC = 1 : 3 \Rightarrow BD = b,\ DC = 3b,\ BC = 4b$
Thus, $b = 3a$.
$BD + DE = BE \Leftrightarrow b + DE = 7a \Leftrightarrow 3a + DE = 7a \Rightarrow DE = 4a$
Therefore, the required ratio is $4a$: $12a$ = **<u>1 : 3</u>**.



Detailed Solutions for GBML Meet 2 - NOVEMBER 2017

ROUND 4

1. $\dfrac{\frac{y-1}{2}}{1+\frac{2y}{3}}=\dfrac{2}{5y}\left(\dfrac{3y}{2}-2\right)\Leftrightarrow\dfrac{3y-3}{6+4y}=\dfrac{3}{5}-\dfrac{4}{5y}=\dfrac{3y-4}{5y}$

   Cross-multiplying, $15y^2-15y=18y-24+12y^2-16y$

   $\Rightarrow 3y^2-17y+24=0$

   We don't really need to find the individual roots, since the coefficients of the equation give us both the sum and the product of the roots.

   The sum is $\frac{17}{3}$ and the product is $\frac{24}{3}$ and the positive difference is $\underline{\mathbf{\frac{7}{3}}}$.

2. $kx^2+4k-1=6kx-2x^2\Leftrightarrow(k+2)x^2-6kx+(4k-1)=0$

   For the roots to be nonreal, the discriminant must be negative.

   $D=(6k)^2-4(k+2)(4k-1)=36k^2-16k^2-28k+8=20k^2-28k+8=4\left(5k^2-7k+2\right)$

   $(5k-2)(k-1)<0\Rightarrow \underline{\mathbf{\frac{2}{5}<k<1}}$.

3. Suppose the roots are r_1 and r_2. Then: $\begin{cases}k=-(r_1+r_2)\\ j=r_1r_2\end{cases}$

   $r_2=r_1+1\Rightarrow k=-(2r_1+1)\Rightarrow r_1=\dfrac{-1-k}{2}$

   Thus, $r_1(r_1+1)=j\Leftrightarrow\left(\dfrac{-1-k}{2}\right)\left(\dfrac{-1-k}{2}+1\right)=-\left(\dfrac{1+k}{2}\right)\left(\dfrac{1-k}{2}\right)=j\Leftrightarrow 4j=k^2-1$

   Since both *j* and *k* are known to positive, we have $\underline{\boldsymbol{k=\sqrt{1+4j}}}$.

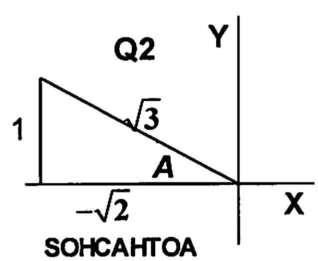
Detailed Solutions for GBML Meet 2 - NOVEMBER 2017

ROUND 5

1.
$$24\tan\theta = 5\cos\theta \Rightarrow 24\frac{\sin\theta}{\cos\theta} = 5\cos\theta$$
$$\Rightarrow 24\sin\theta = 5\cos^2\theta = 5\left(1-\sin^2\theta\right)$$
$$\Rightarrow 5\sin^2\theta + 24\sin\theta - 5 = 0$$
$$\Rightarrow (5\sin\theta - 1)(\sin\theta + 5) = 0 \Rightarrow \sin\theta = \underline{\frac{1}{5}}, \cancel{-5}$$

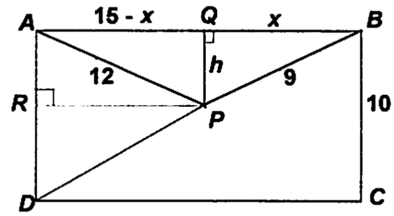
2.
$$\cos x + \sin x = p$$
$$\Rightarrow (\cos x + \sin x)^2 = p^2$$
$$\Rightarrow \cos^2 x + 2\sin x\cos x + \sin^2 x = p^2$$
$$\Rightarrow \left(\sin^2 x + \cos^2 x\right) + (2\sin x\cos x) = p^2 \Rightarrow \sin 2x = p^2 - 1$$
Now
$$\sin^2 2x + \cos^2 2x = \left(p^2-1\right)^2 + q^2$$
$$\Leftrightarrow \left(p^2-1\right)^2 + q^2 = 1$$
$$\Leftrightarrow q^2 = 1 - \left(p^2-1\right)^2 = 2p^2 - p^4 = p^2\left(2-p^2\right)$$
$$\Rightarrow q = \pm p\sqrt{2-p^2}$$
$$p < 0 \text{ and } q > 0 \Rightarrow q = \underline{\mathbf{-p\sqrt{2-p^2}}}.$$
Argue that the maximum value of p is $\sqrt{2}$ and the radicand is always nonnegative.

3.
$$\frac{(\cos 495°)\cot A + \csc A(\tan 210°)}{\tan A\cdot\sec A\cdot\csc x} = \sec 300°$$
$$\Rightarrow \frac{(\cos 135°)\cot A + \csc A(\tan 30°)}{\tan A\cdot\sec A\cdot\csc x} = \sec 60°$$
$$\Rightarrow \frac{-\frac{\sqrt{2}}{2}\cdot -\sqrt{2} + \sqrt{3}\cdot\frac{\sqrt{3}}{3}}{-\frac{1}{\sqrt{2}}\cdot -\frac{\sqrt{3}}{\sqrt{2}}\cdot\csc x} = 2 \Rightarrow \frac{1+1}{\frac{\sqrt{3}}{2}\csc x} = 2$$
$$\Rightarrow \csc x = \frac{2}{\sqrt{3}} \Rightarrow \sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \underline{60°}, \underline{120°}$$



Detailed Solutions for GBML Meet 2 - NOVEMBER 2017

TEAM ROUND



1. $\begin{cases} h^2+(15-x)^2 = 12^2 = 144 \\ h^2 + x^2 = 9^2 = 81 \end{cases}$

Plan: Find *PR*, *RD* and use Pythagorean Theorem on ΔPRD

Subtracting, $225-30x=63 \Rightarrow x=\frac{162}{30}=\frac{54}{10}=5.4$

To avoid the numerical nastiness of the Pythagorean Theorem,

Observe that in ΔPQB, $(\_\_,5.4,9)=\frac{1}{10}(\_\_,54,90)=\frac{18}{10}(\_\_,3,5)$

Since we are dealing with a triangle similar to the 3-4-5, $h=PQ=1.8(4)=7.2$

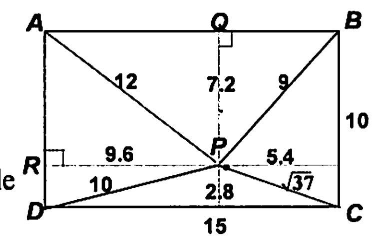
In ΔPQA, $(\_\_,9.6,12)=\frac{1}{10}(\_\_,96,120)=\frac{24}{10}(\_\_,4,5)$

Since we are again dealing with a triangle similar to the 3-4-5, $h=PQ=2.4(3)=7.2$.

In ΔPRD, $(2.8,\ 9.6,\_\_)=\frac{1}{10}(28,96,\_\_)=\frac{4}{10}(7,24,\_\_)$

ΔPRD is similar to a 7-24-25 right triangle, implying $PD=.4(25)=\underline{\mathbf{10}}$.

Check out the diagram at the right. The sum of the squares of the distances from the interior point *P* to the opposite vertices of the rectangle must be equal and $9^2+10^2=12^2+\left(\sqrt{37}\right)^2=181$



2. Given: $\begin{cases} p+q=-3 \\ pq=k \end{cases}$

New sum: $\frac{3}{2p-1}+\frac{3}{2q-1}=\frac{3(2q-1)+3(2p-1)}{(2p-1)(2q-1)}=\frac{6(p+q)-6}{4pq-2(p+q)+1}=\frac{-24}{4k+7}$

New product: $\frac{3}{2p-1}\cdot\frac{3}{2q-1}=\frac{9}{(2p-1)(2q-1)}=\frac{9}{4k+7}$

The new equation is $x^2+\frac{24}{4k+7}x+\frac{9}{4k+7}=0$

Clearing fractions, $(A,B,C)=\underline{\mathbf{(4k+7,24,9)}}$.

TEAM ROUND - continued

3. Given: $\begin{cases} \sin x \cdot \sin 3x \cdot \cos 3x = \left(\sin^2 150°\right) \cdot \sin x \\ \sin x \geq 0 \end{cases}$

$$\sin x\left(\sin 3x \cdot \cos 3x - \left(\frac{1}{2}\right)^2\right) = 0$$

$$\Leftrightarrow \sin x\left(4 \sin 3x \cos 3x - 1\right) = 0$$

$$\Leftrightarrow \sin x\left(2 \sin 6x - 1\right) = 0$$

$$\Leftrightarrow \sin x = 0 \text{ or } \sin 6x = \frac{1}{2}$$

$$\Rightarrow x = \underline{\mathbf{0}}°, \underline{\mathbf{180}}° \text{ or } 6x = \begin{cases} 30° + n \cdot 360° \\ 150° + n \cdot 360° \end{cases} \Leftrightarrow x = \begin{cases} 5° + n \cdot 60° \\ 25° + n \cdot 60° \end{cases}$$

Since $\sin x \geq 0$, solutions must be restricted to quadrants 1 and 2 over $0° \leq x < 360°$.
$n = 0, 1, 2$ gives additional answers in the specified range, **<u>5</u>**°, **<u>65</u>**°, **<u>125</u>**°, **<u>25</u>**°, **<u>85</u>**°,**<u>145</u>**°