GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2018

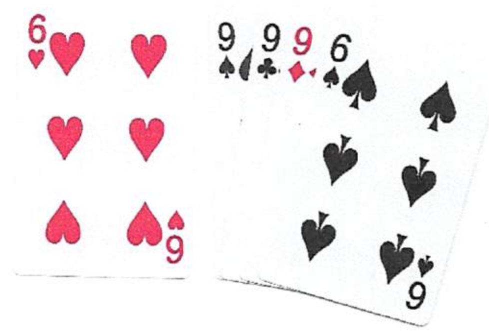
ROUND 1- Arithmetic: Open

1. ______________________________

2. ______________________________

3. ______________________________

1. In the game of cribbage, two points are awarded for any combination of cards that add up to 15. Two points are awarded for any pair (2 cards of the same denomination). Compute the number of points awarded to the following hand of 5 cards:



2. I am thinking of a natural number which has an odd number of factors and is divisible by both 3 and 7. Determine the *minimum* natural number satisfying these conditions.

3. Moe counts by 7s: 1, 8, 15, 22, ...
   Larry counts by 11s: 1, 12, 23, 34, ...
   Curly counts by 13s: 1, 14, 27, 40, ...
   Assuming each boy starts counting with 1, compute the sum of the 4*smallest four-digit* numberscommon to each of these sequences.

Detailed Solutions for GBML Meet 2 - NOVEMBER 2018

ROUND 1

1. Pairs: 6s-♥/♠, 9s -♠/♦, ♠/♣, and ♣/♦ @ 2 points each ⇒ 8 points.
   15-sum: 9 (3 ways) + 6 (2 ways) ⇒ 6 combos @ 2 points each ⇒12 points
   Total: **<u>20</u>** points

2. Only perfect squares have an odd number of factors.
   $3^2 \cdot 7^2 = 9(49) = \underline{\mathbf{441}}$ has $(2+1)(2+1) = 9$ factors.

3. Moe's counting is an arithmetic sequence of the form $7a+1$.
   Larry's counting is an arithmetic sequence of the form $11b+1$.
   Curly's counting is an arithmetic sequence of the form $13c+1$.
   Since $7 \cdot 11 \cdot 13 = 1001$, the 4-digit integers common to each sequence are of the form $1001n+1$.
   $1002 + 2003 + 3004 + 4005 = \underline{\mathbf{10014}}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2018

ROUND 2- Simultaneous Linear equations, Word Problems, Matrices

1. ______________________________

2. ______________________________

3. ____________ : ____________

1. Given: $\begin{cases} a+2b+3c=4 \\ 5b+6c=-7 \\ c=8 \end{cases}$

   Compute $\dfrac{ab+bc+ac}{abc}$.

2. Compute <u>all</u> possible ordered pairs of real numbers for which

   $\begin{bmatrix} x^2-x-5 & y^2-16 & 0 \\ 0 & \dfrac{\sqrt{x^2+y^2}}{5} & 0 \\ 0 & 0 & |x+y| \end{bmatrix}$ is the identity matrix.

3. A bowl contains jelly beans, exclusivelyorange (O)and lemon (L)
   Initially, $O{:}L = 3{:}5$.
   If x jelly beans of each flavor were added, the $O{:}L$ratio would be 7:10.
   If,instead,$y$jelly beans of each flavor were removed, the $O{:}L$ratio would be 5:9.
   Compute the ratio $y : x$.

Detailed Solutions for GBML Meet 2 - NOVEMBER 2018

ROUND 2

1. $\begin{cases} a+2b+3c=4 \\ 5b+6c=-7 \\ c=8 \end{cases} \Rightarrow (a,b,c)=(2,-11,8)$

$$\frac{ab+bc+ac}{abc}=\frac{1}{c}+\frac{1}{a}+\frac{1}{b}=\frac{1}{8}+\frac{1}{2}-\frac{1}{11}=\frac{5}{8}-\frac{1}{11}=\frac{55-8}{88}=\underline{\mathbf{\frac{47}{88}}}.$$

2. Given: $\begin{bmatrix} x^2-x-5 & y^2-16 & 0 \\ 0 & \frac{\sqrt{x^2+y^2}}{5} & 0 \\ 0 & 0 & |x+y| \end{bmatrix}$

Since the 3 x 3 identity matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we have

$x^2-x-5=1 \Rightarrow x^2-x-6=(x-3)(x+2)=0 \Rightarrow x=-2,3$

$y^2-16=0 \Rightarrow y=\pm 4$

$\frac{\sqrt{x^2+y^2}}{5}=1 \Rightarrow x^2+16=25 \Rightarrow x=+3$ only.

Since $|x+y|=1$, the only possible ordered pair is $\underline{\mathbf{(3,-4)}}$.

3. Suppose there were, initially, $3n$ orange jelly beans and $5n$ lemon jelly beans. Then:

$\frac{3n+x}{5n+x}=\frac{7}{10} \Rightarrow 30n+10x=35n+7x \Rightarrow 5n=3x \Rightarrow x=\frac{5}{3}n$

Removing the y jelly beans, we have

$\frac{3n-y}{5n-y}=\frac{5}{9} \Rightarrow 27n-9y=25n-5y \Rightarrow n=2y \Rightarrow y=\frac{1}{2}n.$

Thus, $y:x=\frac{\frac{1}{2}n}{\frac{5}{3}n}=\underline{\mathbf{3:10}}$.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 2 - NOVEMBER 2018

ROUND 3- Geometry: Angles and Triangles

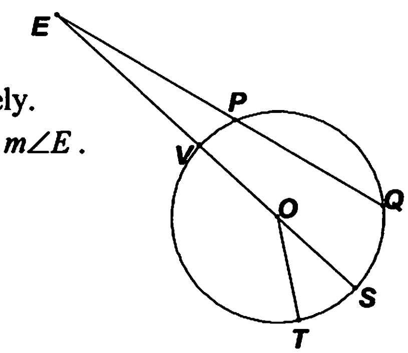
1. ______________________

2. ______________________

3. ______________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. In circle O, let the degree-measures of the consecutive minor arcs $\overset{\frown}{VP}, \overset{\frown}{PQ}, \overset{\frown}{QS}, \overset{\frown}{ST}$ be $2x, 9x, 4x, 3x$, respectively. Points E, V, O, and S are collinear. Compute $m\angle VOT - m\angle E$.

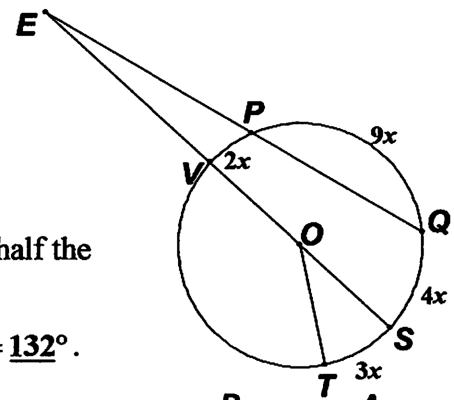


2. ΔABC is a right triangle in which $m\angle A = x-10$, $m\angle B = y+20$, $m\angle C = x+y+k$.
If C is the right angle, $k = k_1$.
If B is the right angle and $k = 8$, then $x = x_1$.
If A is the right angle and $k = 6$, then $y = y_1$.
Compute $k_1 + x_1 + y_1$.

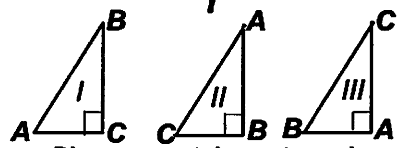
3. Point C is the centroid of equilateral triangle PQR.
Compute the perimeter of ΔPQR, if the perimeter and area of ΔCQR are numerically equal.

Detailed Solutions for GBML Meet 2 - NOVEMBER 2018

ROUND 3

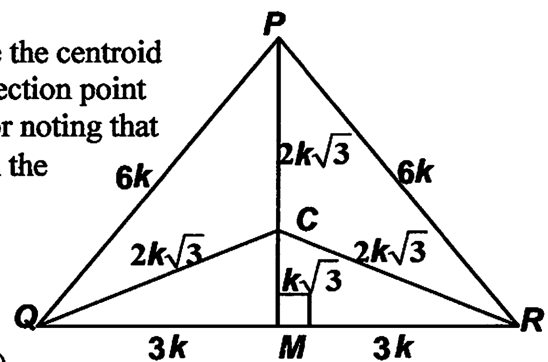


1. $2x+9x+4x=15x=180 \Rightarrow x=12$
   Thus, the degree measure of minor arc $\overparen{VT}$ is $180°-3(12°)=144°$
   Since $\angle VOT$ is a central angle, $m\angle VOT=144°$.
   Since $\angle E$ is formed by two secant lines, its measure is half the difference of its intercepted arcs. Specifically,
   $m\angle E=\frac{1}{2}(4x-2x)=x=12$. Thus, $M\angle VOT-m\angle E=\underline{132}°$.



Diagrams not drawn to scale

2. Given: $m\angle A=x-10,\ m\angle B=y+20,\ m\angle C=x+y+k$
   I:(C = rt. $\angle$) $x+y+k=90=m\angle A+m\angle B=x+y+10 \Rightarrow k=k_1=10$
   II:(B = rt. $\angle$, $k=8$) $y=70,\ m\angle A+m\angle C=2x-10+70+8=90 \Rightarrow x=x_1=11$
   III:(A = rt. $\angle$, $k=6$) $x=100,\ m\angle B+m\angle C=100+2y+20+6=90 \Rightarrow y=y_1=-18$
   Thus, $k_1+x_1+y_1=\underline{3}$.



3. Let the side of the original equilateral triangle be $6k$. Since the centroid (the intersection of the three medians) is located at the trisection point further from the vertex, using the Pythagorean Theorem (or noting that ΔPCQ must be isosceles), gives us the lengths indicated in the diagram at the right.
   Therefore, Per(ΔCQR) = Area(ΔCQR)
   $\Leftrightarrow 6k+4k\sqrt{3}=2k\left(2\sqrt{3}+3\right)=\frac{1}{2}(6k)\left(k\sqrt{3}\right)=3k^2\sqrt{3}$
   $\Rightarrow k=\frac{2\left(2\sqrt{3}+3\right)}{3\sqrt{3}}=\frac{2\sqrt{3}\left(2\sqrt{3}+3\right)}{9}=\frac{12+6\sqrt{3}}{9}=\frac{2\left(2+\sqrt{3}\right)}{3}$
   Thus, $\text{Per}\left(\Delta PQR\right)=18k=\underline{\mathbf{12\left(2+\sqrt{3}\right)}}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2018

ROUND 4- Algebra 2: Quadratic Equations, Problems involving them andTheory of Quadratics

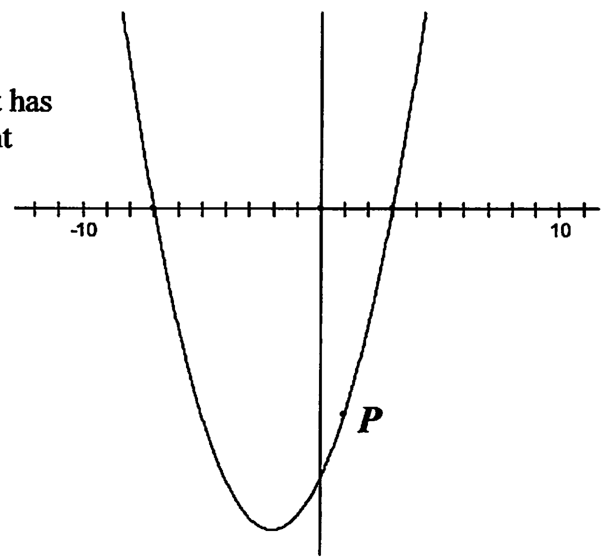
1. (____________, ____________)

2. (____________, ____________)

3. (____________, ____________)

1. The quadratic equation $ax^2+bx+c=0$ has integer roots r_1 and r_2, where $r_1<r_2$, and the coefficients a, b, and c (in some order) are in the ratio 1:2:3. Compute the ordered pair (r_1,r_2).

2. The vertical parabola shown in the graph at the right has x-intercepts as indicated and passes through the point $P(1,-16)$. Compute (h,k), the coordinates of the vertex of this parabola.

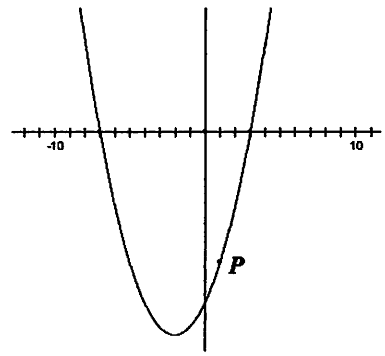


3. The sum of two integersx and y is 78. The arithmetic average of these integers is 3 more than the geometric average of these integers.
If $x<y$, compute the ordered pair (x,y).

Detailed Solutions for GBML Meet 2 - NOVEMBER 2018

ROUND 4

1. Only $a:b:c=1:3:2$ produces integer roots.
$x^2+3x+2=(x+1)(x+2)=0 \Rightarrow x=-1,-2$
$r_1<r_2 \Rightarrow (r_1,r_2)=\underline{\mathbf{(-2,-1)}}$.



2. The x-intercepts are located at $(-7,0)$ and $(3,0)$. Since the parabola is vertical, the equation has the form
$y=f(x)=ax^2+bx+c$.
Substituting the coordinates of the known points, we have
$$\begin{cases} \#1)\ 49a-7b+c=0 \\ \#2)\ \ 9a+3b+c=0 \\ \#3)\qquad a+b+c=-16 \end{cases}$$
$$\Rightarrow \begin{cases} (\#1-\#2): \qquad 4a-b=0 \\ (\#2-\#3)/2: 4a+b=8 \end{cases} \Rightarrow (a,b,c)=(1,4,-21).$$
Completing the squareand noting that the low point on the graph occurs when the squared quantity has a minimum value, i.e., equals zero,
$y=x^2+4x-21=(x+2)^2-25 \Rightarrow V\underline{\mathbf{(-2,-25)}}$.

Alternately, since the zeros of the function are $x=3$ and $x=-7$, the equation can be written as $y=f(x)=a(x-3)(x+7)$. Since the graph contains the point $P(1,-16)$,
$-16=a(-2)(8) \Rightarrow a=1$ and the vertex must be $\left(\frac{3-7}{2},y\right)=(-2,y)$. Substituting,
$f(-2)=-5\cdot 5=-25$ and the vertex must have coordinates $\underline{\mathbf{(-2,-25)}}$.

3. Let $(x,y)=(x,78-x)$, where $x<78-x \Rightarrow x<39$.
$$\frac{x+(78-x)}{2}=3+\sqrt{x(78-x)} \Rightarrow 36^2=x(78-x) \Leftrightarrow x^2-78x+36^2=0.$$
$36^2=36(36)=(4\cdot 9)(6\cdot 6)=(4\cdot 6)(9\cdot 6)=24\cdot 54$ and $24+54=78$.
Thus, $(x,y)=\underline{\mathbf{(24,54)}}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2018

ROUND 5- Trig Equations

1. (______________ , ______________)

2. ______________________________

3. ______________________________

1. Given: $\tan^2(6x) = 3$

   Compute the ordered pair (M, m), where M and m denote the maximum and minimum solutions (in degrees), respectively, over the interval $[0, 90°)$.

2. Given: $2\tan\theta - \cot\theta = -1$

   For θ-values in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, compute <u>all</u> possible values of $\sin\theta$.

3. Given: $\sin x \cdot \sin 2x = 1 - \sin^2 x$

   Compute <u>all</u> possible values of $\cos x$.

Detailed Solutions for GBML Meet 2 - NOVEMBER 2018

ROUND 5

1. $\tan^2(6x)=3 \Rightarrow \tan(6x)=\pm\sqrt{3} \Rightarrow 6x=\left.\begin{matrix}60^\circ\\120^\circ\end{matrix}\right\}+n(180^\circ) \Leftrightarrow x=\left.\begin{matrix}10^\circ\\20^\circ\end{matrix}\right\}+n(30^\circ)$

   Since $x\in[0^\circ,90^\circ)$, $n=0$, 1, 2 gives us the 6 solutions over the specified interval, namely, $10^\circ, 40^\circ, 70^\circ, 20^\circ, 50^\circ, 80^\circ$. Thus, $(M,m)=(80,10)$.

2. All values of θ must be in quadrants 2 or 3. Over the open interval $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, a solution of $\theta=\pi$ would be extraneous, since $\cot\theta$ would be undefined.

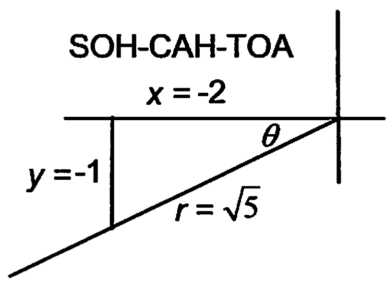
   $2\tan\theta-\cot\theta=-1 \Leftrightarrow 2\tan\theta-\frac{1}{\tan\theta}=-1$.

   $\Rightarrow 2\tan^2\theta+\tan\theta-1=(2\tan\theta-1)(\tan\theta+1)=0$

   $\Rightarrow \tan\theta=\frac{1}{2},-1$. [Check: $2\left(\frac{1}{2}\right)-2=-1$ and $2(-1)-(-1)=-1$]

   As a special value, $\tan\theta=-1 \Rightarrow \theta=\frac{3\pi}{4}$ (quadrant 2) and $\sin\frac{3\pi}{4}=\underline{\frac{\sqrt{2}}{2}}$.

   For $\tan\theta=\frac{1}{2}$, finding an exact *numerical* value for θ is impossible, but computing an exact value (in terms of radicals) for $\sin\theta$, where θ is in quadrant 3, is easy!

   $\sin\theta=\frac{-1}{\sqrt{5}}=\underline{-\frac{\sqrt{5}}{5}}$.

3. $\sin x\cdot\sin 2x=1-\sin^2 x \Leftrightarrow \sin x(2\sin x\cos x)=\cos^2 x$

   $\Leftrightarrow 2\sin^2 x\cos x=\cos^2 x \Leftrightarrow 2(1-\cos^2 x)\cos x=\cos^2 x$

   $\Leftrightarrow 2\cos^3 x+\cos^2 x-2\cos x=0 \Leftrightarrow \cos x(2\cos^2 x+\cos x-2)=0$.

   Thus, $\cos x=\underline{\mathbf{0}}$, and using the quadratic formula, any other values can be found.

   $\cos x=\frac{-1\pm\sqrt{1+16}}{4}=\underline{\frac{\sqrt{17}-1}{4}}$. ($\cos x=\frac{-1-\sqrt{17}}{4}<-1$ is extraneous.)

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2018

TEAM ROUND

3 pts. 1. ____________________

3 pts. 2. (__________ , __________)

4 pts. 3. ____________________ feet

1. Consider a segment on the number line from 0 to 24 to be a seesaw with a fulcrum (i.e., a balance point) at 9. Assume the weights of the 4 members of the Smith family and their locations on the seesaw are Trey (60, 1), Jaden (80, 0), Jada (100, 5), Will (120, 24). Willow, the 5th member of the Smith family weighs 40 lbs. If Willow sits at location x, the seesaw is in equilibrium, i.e., balanced. Compute x.

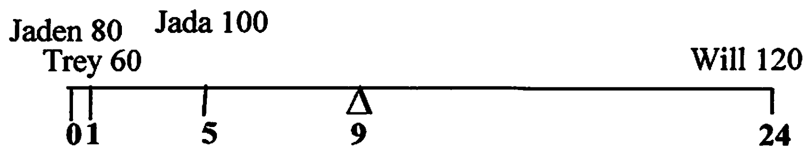
2. The angles of ΔABC have measures of 140_b, 244_b, and 221_b.
If $b > 0$, the *largest* angle in ΔABC has measure $M°$.
If $b < 0$, the *largest* angle in ΔABC has measure m °.
Compute the ordered pair (M, m).

3. A relief convoy of heavily loaded trucks is traveling down the unpaved highway at a constant speed of 15 mph. A jeep traveling at a constant speed of 35 mph patrols the column by riding from the front to the rear of the column and back. If a single roundtrip takes the jeep 3 minutes and 30 seconds, compute the length of the truck convoy in feet. (Note: 1 mile = 5280 feet)

Detailed Solutions for GBML Meet 2 - NOVEMBER 2018

TEAM ROUND

1.



Torque (tendency to turn) is a force equal to the product of a mass and its distance from the pivot point. To maintain equilibrium, the clockwise (CW) and counterclockwise (CCW) torques (measured in ft-lbs.) must be equal.
Jaden, Trey and Jada contribute to CCW torque: $9(80)+8(60)+4(100)=1600$ ft-lbs.
Will contributes to CW torque: $15(120)=1800$ foot-lbs.
Thus, Willow must contribute 200 foot-lbs. of CCW torque.
Since $40\cdot 5=200$, Willow must sit 5 units left of the pivot point, i.e., $x=\underline{\mathbf{4}}$.

2. $140_b+244_b+221_b=\left(b^2+4b\right)+\left(2b^2+4b+4\right)+\left(2b^2+2b+1\right)=180$

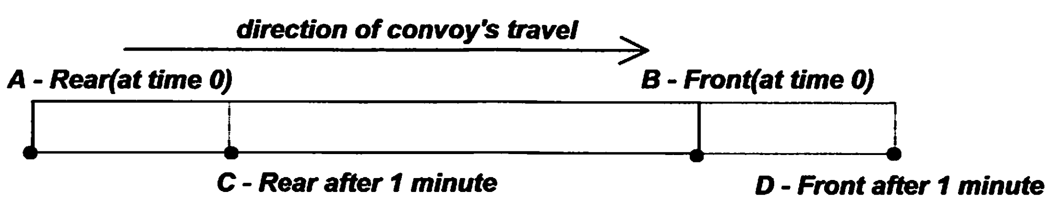
$\Rightarrow 5b^2+10b+5=180 \Rightarrow b^2+2b-35=(b-5)(b+7)=0 \Rightarrow b=5,-7$.

$$b=5\Rightarrow\begin{cases}5^2+4(5)=45\\ 2\left(5^2\right)+4(5)+4=74\\ 2\left(5^2\right)+2(5)+1=61\end{cases}\qquad b=-7\Rightarrow\begin{cases}(-7)^2+4(-7)=21\\ 2(-7)^2+4(-7)+4=74\\ 2(-7)^2+2(-7)+1=85\end{cases}$$

Thus, $(M,m)=\underline{\mathbf{(74,85)}}$.

3. When the jeep is traveling towards the back of the convoy, its effective speed is 50 mph. (35 + 15)
When the jeep is traveling towards the front of the convoy, its effective speed is 20 mph.(35 – 15)
Let T_{back} represent the time (in minutes) to reach the back of the convoy, and let T_{front}represent the time to return to the front of the convoy.

Thus, $\frac{T_{back}}{T_{front}}=\frac{2}{5}$ and $T_{back}+T_{front}=3.5 \Rightarrow T_{\text{back}}=1$ and $T_{\text{front}}=2.5$



In 1 minute, the front of the convoy moves from B to D – a distance of $15\cdot\frac{1}{60}=0.25$ mi.

In 1 minute, the jeep moves from B to C – a distance of $35\cdot\frac{1}{60}=\frac{7}{12}$ mi.

Thus, the length of the convoy is $\frac{7}{12}+\frac{1}{4}=\frac{10}{12}=\frac{5}{6}$ mile = $\frac{5}{6}(5280)$ = 5(880) = **<u>4400</u>** feet.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 - NOVEMBER 2018

Answer Sheet

Round 1

1. 20
2. 441
3. 10014

Round 2

1. $\frac{47}{88}$
2. $(3,-4)$
3. 3 : 10

Round 3

1. 132
2. 3
3. $12(\sqrt{3}+2)$

Round 4

1. $(-2,-1)$
2. $(-2,-25)$
3. $(24,54)$

Round 5

1. (80, 10)
2. $\frac{\sqrt{2}}{2},-\frac{\sqrt{5}}{5}$ (Order is irrelevant.)
3. $0, \frac{\sqrt{17}-1}{4}$ (Order is irrelevant.)

Team Round

1. 4 (3 pts)
2. (74, 85) (3 pts)
3. 4400 (4 pts)