GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2014

ROUND 1 – Algebra 1: Fractions and Word Problems

1. ______________________________

2. ______________________________

3. Smallest: ______ Largest: ________

1. If $x = \frac{a}{b}$ and $b \neq 0$, express $\frac{a+b}{a-b}$ in terms of x as a fraction in simplest form.

2. Twelve quarts of a 70% antifreeze mixture fills a truck's radiator completely. 50% of the antifreeze mixture in a full radiator is drained off and replaced with a 40% antifreeze mixture. Then *N* quarts of this mixture in the radiator is drained off and replaced with an 80% antifreeze mixture. The mixture in the radiator is now 65% antifreeze. Determine the value of *N*.

3. $\frac{10x+y}{10y+x} = \frac{a}{b}$, where x, y, a, and b are positive integers.

   If $x+y=9$, $a+b=11$ and $\frac{a}{b}<1$, compute the largest and smallest values of $10x+y$.

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 2 – Coordinate Geometry of the Straight Line

1. ______________________

2. ______________________

3. ______________________

1. Find <u>all</u> values for K for which the line whose equation is $4x - Ky = 13K$ contains the point $\left(\frac{45}{2}, K\right)$.

2. Given: $P(-8, J-1)$, $Q(1, -2)$ and $R(5K+6, -12)$
   Compute <u>all</u> possible ordered pairs (J, K) for which the product of the slopes of $\overline{PQ}$ and $\overline{QR}$ is $+1$, and the slopes of $\overline{PR}$ and $\overline{QR}$ are equal.

3. The line L_1 passes through points $A(3,8)$ and $B(9,4)$. The equation of the perpendicular bisector of $\overline{AB}$ is $px + qy + w = 0$, where p, q and w are integers and $p > 0$. The shortest distance from $C(1,5)$ to L_1 is K units. Compute the ordered pair (w, K).

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ROUND 3 – Geometry – Polygons: Area and Perimeter

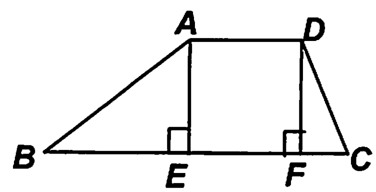
1. ______________________

2. ______________________

3. ______________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. Trapezoid $ABCD$ has an area of 72 square units. If $BE : EF : FC = 4:3:2$ and $BC = 24$, compute AE.



2. The lengths of the sides of a decagon are consecutive multiples of 3 and the length of its shortest side is 9 units. The longest side of a similar decagon has length 16 units. The perimeter of the smaller decagon is P. Find the value of P.

3. <u>How many</u> non-congruent <u>scalene</u> triangles can be constructed with sides of integral lengths and perimeter less than 15?

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ROUND 4 – Algebra 2: Logs, Exponents Radicals and equations involving them

1. ______________________________

2. ______________________________

3. ______________________________

1. Compute the numerical value of $\left[\dfrac{\left(\dfrac{3}{4}\right)^{3}}{\left(\dfrac{4}{25}\right)^{-3/2}}\right]^{-2/3} + 8\left(\sqrt{27}\right)^{-4/3}$

2. Find <u>all</u> real values of x for which $x^{3/2} - 16x^{7/10} = 0$.

3. Compute x if $\log_2\left(x^2\right) - \log_9 3 + \log_{\sqrt[3]{3}} 9 = \log_4\left(\dfrac{1}{8}\right) - \log_x 64$.

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 5 – Trig Analysis and Complex Numbers in Trig Form

1. ______________________________

2. ______________________________

3. ______________________________

1. Compute $\dfrac{\sqrt{2}cis27°}{cis72°}$ Express your answer in rectangular form.

2. The <u>number</u> of distinct solutions of the equation $\left(2\sin 2\theta - \sqrt{3}\right)\left(\cos 3\theta - \sin 180°\right) = 0$ over the interval $0° \le x < 360°$ is *K*. Find the value of *K*.

3. Solve for *x*, $0 \le x < 360°$: $1 - \tan x = \sqrt{2}\sec x$

GREATER BOSTON MATHEMATICS LEAGUE

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TEAM ROUND

3 pts.	1.	____________
3 pts.	2.	____________
4 pts.	3.	____________

1. Points A $(x, 7)$ and B $(15, y)$ are the endpoints of line segment $\overline{AB}$.
   Point P $(8, -11)$ is the trisection point on $\overline{AB}$ nearer to point B.
   Compute $x + y$.

2. Given triangle ABC, point D lies on $\overline{BC}$ such that $\overline{AD}$ is an angle bisector.
   If $AD = 10$, $AB = 16$, $AC = 9$, compute BC.

3. $\sum_{k=13}^{2002}\left[\log\left(k^2-9\right)-\log\left(k^2-4\right)\right]=\log\left(\frac{a}{b}\right)$, where a and b are relatively prime integers.
   Compute $a + b$.

GREATER BOSTON MATHEMATICS LEAGUE

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Answer Sheet

Round 1

1. $\frac{x+1}{x-1}$
2. 24/5 or 4.8
3. Smallest: 18 Largest: 45

Round 2

1. $-18, 5$
2. $(-10, -3), (8, 1)$
3. $\left(-6, \sqrt{13}\right)$

Round 3

1. 4.5 or $\frac{9}{2}$
2. 100
3. 6

Round 4

1. 12
2. 0, 32
3. $\frac{1}{4}, \frac{\sqrt{2}}{4}$

Round 5

1. $1-i$
2. 8
3. 315°

Team Round

1. -26 (3 pts)
2. $\frac{25\sqrt{11}}{6}$ (3 pts)
3. 1001 (4 pts)

Detailed Solutions for GBML Meet 3 – DECEMBER 2014

ROUND 1

1. $a = bx \Rightarrow \dfrac{a+b}{a-b} = \dfrac{bx+b}{bx-b} = \dfrac{\not{b}(x+1)}{\not{b}(x-1)} = \underline{\dfrac{\mathbf{x+1}}{\mathbf{x-1}}}$, since $b \neq 0$.

2. 6(.70) + 6(.4) ⇒ 6.6/12 = 11/20 ⇒ 55% antifreeze mixture
(12 – N)(.55) + .8N = 12(.65) ⇒ 6.6 + .25N = 7.8 ⇒ N = 4(1.2) = **4.8** or **24/5**

3. $\dfrac{10x+y}{10y+x} = \dfrac{a}{b} \Rightarrow 10bx - ax = 10ay - by \Rightarrow \dfrac{x}{y} = \dfrac{10a-b}{10b-a}$

Since $a+b=11$, $\dfrac{a}{b}<1$ and a and b are positive integers, the possible ordered pairs (a,b) are $(1,10)$, $(2,9)$, $(3,8)$, $(4,7)$, $(5,6)$.

The first ordered pair is rejected, since it implies that $x=0$.

$(2,9) \Rightarrow \dfrac{11}{88} = \dfrac{1}{8} \Rightarrow 10x + y = \underline{\mathbf{18}}$ (smallest)

$(3,8) \Rightarrow \dfrac{22}{77} = \dfrac{2}{7} \Rightarrow 10x + y = 27$

$(4,7) \Rightarrow \dfrac{33}{66} = \dfrac{3}{6} \Rightarrow 10x + y = 36$

$(5,6) \Rightarrow \dfrac{44}{55} = \dfrac{4}{5} \Rightarrow 10x + y = \underline{\mathbf{45}}$ (largest)

Detailed Solutions for GBML Meet 3 – DECEMBER 2014

ROUND 2

1. Since the coordinates of any point on a line must satisfy the equation of the line, we have

$$4\left(\frac{45}{2}\right)-K(K)=13K \Leftrightarrow K^2+13K-90=0$$
$$\Leftrightarrow (K+18)(K-5)=0$$
$$\Leftrightarrow K=\underline{\mathbf{-18,\ 5}}$$

2. Slopes of $\overline{PQ}$ and $\overline{QR}$: $\frac{J+1}{-9}\cdot\frac{-10}{5K+5}=1 \Leftrightarrow \frac{J+1}{9}=\frac{K+1}{2} \Rightarrow 2J=9K+7$

Slopes of $\overline{PR}$ and $\overline{QR}$: $\frac{-5K-14}{J+11}=\frac{5K+5}{-10}=\frac{K+1}{-2} \Rightarrow 2(5K+14)=(K+1)(J+11)$

Doubling both sides to obtain $2J$ on the right side,

$20K+56=(K+1)(2J+22)=(K+1)(9K+29)=9K^2+38K+29$

$\Leftrightarrow 9K^2+18k-27=0 \Leftrightarrow 9\left(K^2+2K-3\right)=0 \Leftrightarrow 9(K+3)(K-1)=0$

$K=-3 \Rightarrow \underline{\mathbf{(-10,-3)}}$; $K=1 \Rightarrow \underline{\mathbf{(8,1)}}$ Both ordered pairs are required.

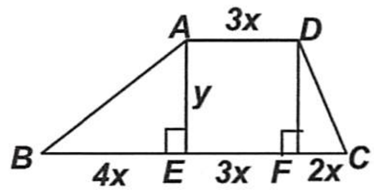
3. The slope $m_{AB}=\frac{8-4}{3-9}=-\frac{2}{3} \Rightarrow m_{\perp}=+\frac{3}{2}$. The midpoint of $\overline{AB}$ is $\left(\frac{3+9}{2},\frac{8+4}{2}\right)=(6,6)$.

Thus, the equation of the perpendicular bisector is $(y-6)=\frac{3}{2}(x-6) \Leftrightarrow 3x-2y-6=0$.

Using the point-to-line distance formula, the distance from $P(u,v)$ to $Ax+By+C=0$ is $\frac{|Au+Bv+C|}{\sqrt{A^2+B^2}}$. Therefore, $K=\frac{|3\cdot1-2\cdot5-6|}{\sqrt{3^2+2^2}}=\frac{13}{\sqrt{13}}=\sqrt{13}$ and $(w,K)=\underline{\mathbf{(-6,\sqrt{13})}}$.

Detailed Solutions for GBML Meet 3 – DECEMBER 2014

ROUND 3



1. Since $AEFD$ must be rectangle $AD = EF = 3x$ and the area of trapezoid $ABCD$ is $\frac{1}{2}y(3x+9x) = 72 \Rightarrow xy = 12$.

   But $BC = 9x = 24 \Rightarrow x = \frac{8}{3} \Rightarrow y = 12 \cdot \frac{3}{8} = \underline{\mathbf{4.5}}$ or $\underline{\mathbf{\frac{9}{2}}}$.

2. The 10 sides in the original decagon have lengths 9, 9 +1(3), 9 + 2(3), … 9 + 9(3) = 36, resulting in a perimeter of 9 + 12 + 15 + … + 30 + 33 + 36 = 5(45) = 225
   If the longest side in the similar decagon has length 16, the proportionality constant is $\frac{16}{36} = \frac{4}{9}$. Thus, the perimeter of this similar decagon is $\frac{4}{9} \cdot 225 = 4 \cdot 25 = \underline{\mathbf{100}}$.

3. Consider the following chart of possible side lengths for **<u>scalene</u>** triangles with integer side lengths and perimeters less than 15. Only scalene triangles are highlighted.

Per					
3	111				
4	None				
5	122				
6	222				
7	133	223			
8	233				
9	333	234	144		
10	334	244			
11	335	344	245	155	
12	444	345	255		
13	445	355	346	256	166
14	554	446	356	266	

Thus, there are **<u>6</u>** possible triangles.

Detailed Solutions for GBML Meet 3 – DECEMBER 2014

ROUND 4

1. $$\left[\frac{\left(\frac{3}{4}\right)^3}{\left(\frac{4}{25}\right)^{-3/2}}\right]^{-2/3}+8\left(\sqrt{27}\right)^{-4/3}=\frac{\left(\frac{3}{4}\right)^{-2}}{\left(\frac{4}{25}\right)^{1}}+8\cdot 3^{\frac{3}{2}\cdot\frac{-4}{3}}=\frac{16}{9}\cdot\frac{25}{4}+8\cdot 3^{-2}=\frac{108}{9}=\underline{\mathbf{12}}$$

2. Clearly, $x=0$ is a solution. For $x\neq 0$,
$x^{3/2}=16x^{7/10}\Leftrightarrow x^{3/2}\cdot x^{-7/10}=16\Leftrightarrow x^{3/2-7/10}=16\Leftrightarrow x^{4/5}=2^4\Leftrightarrow x^{1/5}=2\Leftrightarrow x=32$
Thus, the only real solutions are **<u>0, 32</u>**.

3. $\log_2\left(x^2\right)-\log_9 3+\log_{\sqrt[3]{3}} 9=\log_4\left(\frac{1}{8}\right)-\log_x 64$

$\Leftrightarrow 2\log_2 x-\frac{1}{2}+6=-\frac{3}{2}-6\log_x 2$

$\Leftrightarrow 2\log_2 x+6\log_x 2+7=0$

$\Leftrightarrow 2\log_2 x+6\left(\frac{1}{\log_2 x}\right)+7=0$

$\Leftrightarrow 2\left(\log_2 x\right)^2+7\log_2 x+6=0$

$\Leftrightarrow \left(2\log_2 x+3\right)\left(\log_2 x+2\right)=0\Rightarrow \log_2 x=-\frac{3}{2},-2$

$\Rightarrow x=2^{-2}=\underline{\mathbf{\frac{1}{4}}}$

$\Rightarrow x=2^{-3/2}=\left(\frac{1}{\sqrt{2}}\right)^3=\frac{1}{2\sqrt{2}}=\underline{\mathbf{\frac{\sqrt{2}}{4}}}$

Detailed Solutions for GBML Meet 3 – DECEMBER 2014

ROUND 5

1. $\dfrac{\sqrt{2}cis27^\circ}{cis72^\circ} = \sqrt{2}cis\left(27^\circ - 72^\circ\right) = \sqrt{2}cis\left(-45^\circ\right)$

$= \sqrt{2}\left(\cos 45^\circ - i\sin 45^\circ\right) = \sqrt{2}\left(\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}\right) = \underline{\mathbf{1 - \mathit{i}}}$

2. $\sin 2\theta = \dfrac{\sqrt{3}}{2} \Rightarrow 2\theta = \left.\begin{matrix} 60^\circ \\ 120^\circ \end{matrix}\right\} + k\left(360^\circ\right) \Rightarrow \theta = \left.\begin{matrix} 30^\circ \\ 60^\circ \end{matrix}\right\} + k\left(180^\circ\right)$ - 4 solutions

   $\cos 3\theta = 0 \Rightarrow 3\theta = 90^\circ + k\left(180^\circ\right) \Rightarrow \theta = 30^\circ + k\left(60^\circ\right)$ - 6 solutions

   Discounting the overlap ($\left(\left(30^\circ, 210^\circ\right)\right.$, there are **<u>8</u>** distinct solution over $0^\circ \le x < 360^\circ$.

3. $1 - \tan x = \sqrt{2}\sec x \Rightarrow x \ne 90^\circ + n\left(180^\circ\right)$ (otherwise both $\tan x$ and $\sec x$ are undefined).

   Multiplying by $\cos x$, we have $\cos x - \sin x = \sqrt{2}$

   Squaring both sides, $\cos^2 x - 2\sin x\cos x + \sin^2 x = 2$

   $\Rightarrow -2\sin x\cos x = 1$

   $\Rightarrow \sin 2x = -1$

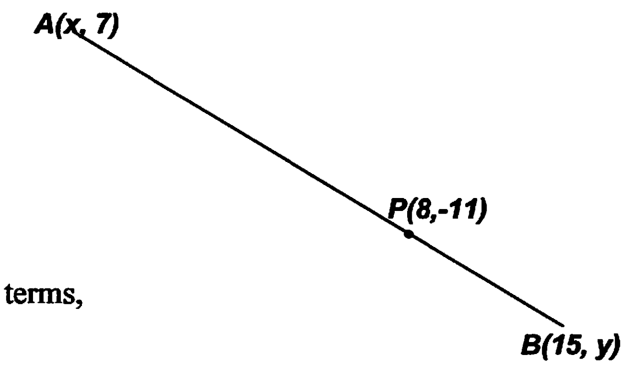
   $\Rightarrow 2x = 270^\circ + n\left(360^\circ\right)$

   $x = 135^\circ + n\left(180^\circ\right) \Rightarrow x =$ ~~135°~~, **<u>315°</u>**

   Squaring both sides can introduce extraneous solutions, so checking these solutions is a must. 135° is rejected, since the left side of the original equation evaluates to +2 and the right side evaluates to –2.

Detailed Solutions for GBML Meet 3 – DECEMBER 2014

TEAM ROUND



1. Equating slopes, $\frac{y+11}{7}=\frac{18}{x-8}\Rightarrow y+11=\frac{126}{x-8}$

$PB=2PA\Rightarrow 4\left(49+(y+11)^2\right)=(8-x)^2+18^2$

Substituting, $196+4\left(\frac{126}{x-8}\right)^2=(8-x)^2+324$

Multiplying through by $(x-8)^2$ and transposing terms,

$(x-8)^4+128(x-8)^2-4(126)^2=0$

Notice $4(126)^2=4\cdot(2\cdot 63)^2=(4\cdot 7^2)(4\cdot 9^2)=(196)(324)$ and these factors differ by 128

Thus, the expression on the left side of the equation factors as

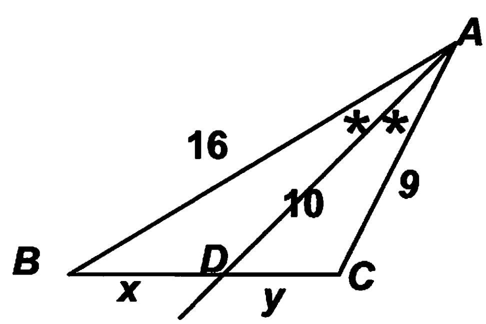
$\left((x-8)^2-196\right)\underline{\left((x-8^2)+324\right)}=0\Rightarrow (x-8)^2=196=14^2$ and the underlined expression is always positive. $x-8=\pm 14\Rightarrow (x,y)=(22,-2),\ (-6,-20)$. From the diagram, we see that $(22,-2)$ is extraneous and the only possible ordered pair is $(-6,-20)\Rightarrow x+y=\underline{-26}$.

Alternately (and much easier), we write a "mass point" equation or invoke a weighted average.

$P=\frac{A+2B}{3}\Rightarrow 3(8,1)=(x,7)+2(15,y)\Rightarrow (24,3)=(x+30,7+2y)\Rightarrow (x,y)=(-6,-20)\Rightarrow \underline{-26}$.

2. Using the angle bisector theorem,

$\frac{x}{16}=\frac{y}{9}\Rightarrow\frac{x}{y}=\frac{16}{9}$. Let $x=16a$ and $y=9a$.



Using Stewart's theorem,

$16^2\cdot 9a+9^2\cdot 16a=10^2(25a)+25a\cdot 16a\cdot 9a$

Pulling out common factors on each side,

$16\cdot 9a(16+9)=(25a)(10^2+144a^2)$

Dividing through by $25a$ (since $a\neq 0$),

$16\cdot 9=100+144a^2\Rightarrow a^2=\frac{44}{144}=\frac{11}{36}\Rightarrow a=\frac{\sqrt{11}}{6}\Rightarrow BC=\underline{\frac{25}{6}\sqrt{11}}$.

Detailed Solutions for GBML Meet 3 – DECEMBER 2014

TEAM ROUND - continued

3. $$\sum_{k=13}^{2002}\left[\log\left(k^2-9\right)-\log\left(k^2-4\right)\right]=\sum_{k=13}^{2002}\log\frac{\left(k^2-9\right)}{k^2-4}=$$

$$=\log\left(\frac{13^2-9}{13^2-4}\cdot\frac{14^2-9}{14^2-4}\cdot\frac{15^2-9}{15^2-4}\cdot\ldots\cdot\frac{2002^2-9}{2002^2-4}\right)$$

$$=\log\left(\frac{\cancel{16}\cdot 10}{15\cdot\cancel{11}}\cdot\frac{\cancel{17}\cdot\cancel{11}}{\cancel{16}\cdot\cancel{12}}\cdot\frac{\cancel{18}\cdot\cancel{12}}{\cancel{17}\cdot\cancel{13}}\cdot\frac{\cancel{19}\cdot\cancel{13}}{\cancel{18}\cdot 12}\cdot\frac{20\cdot 14}{\cancel{19}\cdot 13}\cdot\ldots\cdot\frac{2005\cdot 1999}{2004\cdot 2000}\right)$$ [First and last survivors]

$$=\log\left(\frac{10}{15}\frac{2005}{2000}\right)=\log\left(\frac{401}{3\cdot 200}\right)=\log\left(\frac{401}{600}\right)\Rightarrow a+b=\underline{\mathbf{1001}}.$$