# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 3 - DECEMBER 2015

**ROUND 1** - Algebra 1: Fractions and Word Problems

1. ______________________________

2. Bob: $________ Amy: $________

3. (____________ , ____________)

1. Simplify completely: $\left(\dfrac{a}{2a-1}-\dfrac{1}{a}\right)\div\left(1-\dfrac{a}{2-\dfrac{1}{a}}\right)$

2. Initially, Bob has three quarters as much money as Amy, but after the following back and forth transfers (in the given order), they have the same amount of money.
   - Amy gives Bob half of her money.
   - Bob gives Amy three-eighths of the money he now has.
   - Amy gives Bob $6 from the money she now has.

   How much did each of them have initially?

3. Dick sets out on an 8 mile walk. After walking at his usual speed for the first 2 miles, he increases his speed by $\dfrac{1}{4}$ mph for the remaining distance. This resulted in his cutting 6 minutes off his usual time. His usual time for the 8 mile walk is $H$ hours and $M$ minutes. Compute the ordered pair $(H,M)$.

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 3 - DECEMBER 2015

**ROUND 2** - Coordinate Geometry of the Straight Line

1. (_______, _______, _______)

2. (__________, __________)

3. ____________________

1. An equation for the set of points in a plane equidistant from the intercepts for the line $L_1 : 4x - y - 8 = 0$ is $Ax + By + C = 0$. If $A$, $B$, and $C$ are relatively prime positive integers, compute the ordered triple $(A, B, C)$.

2. Line $\mathcal{L}$ passes through point $A$ on the $x$-axis, point $B$ on the $y$-axis and point $C(6, 2)$. The $x$-intercept is three times the $y$-intercept. Point $P(h, k)$ lies on $\overline{AB}$ such that $AP : PB = 1:3$. Compute the ordered pair $(h, k)$.

3. Given: $A(3,1)$, $B(7,5)$, $C(13,-1)$
   In $\Delta ABC$, compute the sum of the lengths of the altitude and median from vertex $A$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2015

**ROUND 3** - Geometry - Polygons: Area and Perimeter

1. ______________________

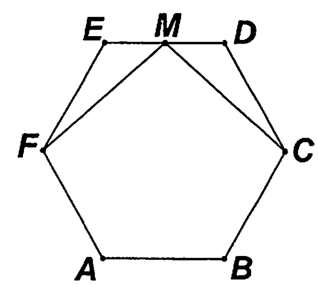
2. ______________________

3. ______________________

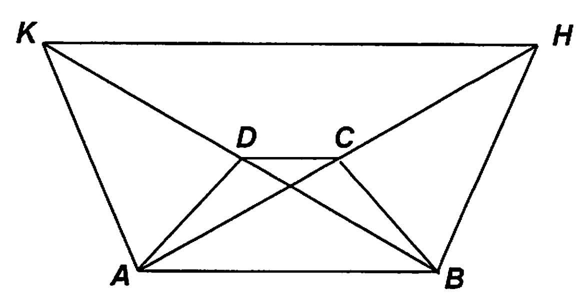
**DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.**

1. An equilateral triangle and a rhombus have the same area. The altitude of the triangle has length $4\sqrt{3}$, and the perimeter of the rhombus is $24\sqrt{3}$. Compute the length of an altitude in the rhombus.

2. Given the regular hexagon $ABCDEF$ with $AB = 1$ and $M$ the midpoint of $\overline{ED}$. Compute the area of pentagon $ABCMF$.




3. In isosceles trapezoid $ABCD$, $AB = 7$ and $DC = 3$. The distance between $\overline{AB}$ and $\overline{DC}$ is 4. $\overline{AC}$ is extended to $H$, so that $CH = AC$. $\overline{BC}$ is extended to $K$, so that $DK = BD$. Compute the area of $ABHK$.



# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 3 - DECEMBER 2015

**ROUND 4** - Algebra 2: Logs, Exponents Radicals and equations involving them

1. (____________ , ____________)

2. ________________________

3. ________________________

1. Compute the ordered pair $(x, y)$ which makes the following statement true:

$$\left(\sqrt{3 \cdot 5^{1/3}}\right)^6 \cdot \sqrt[3]{3^{1/2} \cdot 5^{1/4}} = 15^x \cdot 3^y$$

2. Solve for $x$. $\log_2\left(\log_3(6x)\right) = 1 - \log_a\left(\frac{1}{a}\right)$

3. For what value(s) of $x$ is the following statement true?

$$x = \left(\log_2 5 + \log_4 0.2\right)\left(\log_5 2 + \log_{25} 0.5\right)$$

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 3 - DECEMBER 2015

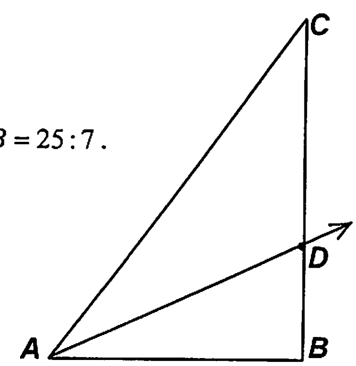
**ROUND 5** - Trig Analysis and Complex Numbers in Trig Form

1. ______________________

2. ______________________

3. ______________________

1. If $\tan x + \cot x = 4$ and $0 \le x < 360°$, compute $(\sin x)(\cos x)$.




2. In $\triangle ABC$, $\measuredangle B$ is a right angle, $\overline{AD}$ bisects $\measuredangle BAC$ and $CD:DB = 25:7$. Compute $\tan C$.

3. If $0 \le x < 360°$, compute the value(s) of $x$ that make the following statement true:

$$\sin 2x - \sin x + 1 = \cos 2x + \cos x$$

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 3 - DECEMBER 2015

### TEAM ROUND

3 pts. 1. (______, ______, ______)

3 pts. 2. __________________

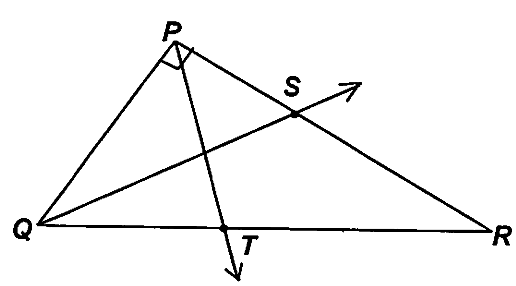
4 pts. 3. __________________

1. In the land of Fanta, the Zoota tribesmen trade using whole coconuts for money. They will trade two spears for three fish hooks and a knife. Three spears, two knifes and one fish hook together cost 25 coconuts. Compute the ordered triple $(S,F,K)$, the individual price (in coconuts) of each of the three items.

2. Compute the value(s) of $x$ over $0 \le x < 360°$ for which the following statement is true:

$$\log_2\left(\frac{2}{\sqrt{\cos x}}\right) = 2 - \frac{1}{2}\log_{\sec x}(2)$$

3. Given: $\overline{PT}$ bisects $\measuredangle P$, $\overline{QS}$ bisects $\measuredangle Q$, $m\measuredangle QPR = 90°$, $QT = 3$, $TR = 4$
Compute $QS$.



## GREATER BOSTON MATHEMATICS LEAGUE

## MEET 3 - DECEMBER 2015

# Answer Sheet

Round 1

1. $-\frac{1}{a}$
2. A: \$64   B: \$48
3. $(2,8)$

Round 2

1. $(1,4,15)$
2. $(9,1)$
3. $9\sqrt{2}$

Round 3

1. $\frac{8}{3}$
2. $\frac{5\sqrt{3}}{4}$
3. 80

Round 4

1. $\left(\frac{13}{12},\frac{25}{12}\right)$
2. $\frac{27}{2}$
3. $\frac{1}{4}$

Round 5

1. $\frac{1}{4}$
2. $\frac{7}{24}$
3. 30°, 150°, 135°, 315°

Team Round

1. $(5,2,4)$   (3 pts)
2. 60°, 300°   (3 pts)
3. $\frac{21\sqrt{5}}{10}$   (4 pts)

## Detailed Solutions for GBML Meet 3 - DECEMBER 2015

### ROUND 1

1. $\left(\frac{a}{2a-1}-\frac{1}{a}\right)\div\left(1-\frac{a}{2-\frac{1}{a}}\right)\Leftrightarrow\left(\frac{a^2-(2a-1)}{a(2a-1)}\right)\div\left(1-\frac{a^2}{2a-1}\right)\Leftrightarrow\left(\frac{(a-1)^2}{a(2a-1)}\right)\div\left(\frac{2a-1-a^2}{2a-1}\right)$

$\Leftrightarrow \frac{(a-1)^2}{a(2a-1)}\div\frac{-(a-1)^2}{(2a-1)}\Leftrightarrow\frac{\cancel{(a-1)^2}}{a\cancel{(2a-1)}}\cdot\frac{\cancel{(2a-1)}}{-\cancel{(a-1)^2}}=\underline{-\frac{1}{a}}$.

2. Let $(A,B)$, where $B=\frac{3}{4}A$, denote the initial holdings of Al and Bob respectively.
Then: $\frac{5}{8}\left(B+\frac{A}{2}\right)+6=\left(\frac{A}{2}+\frac{3}{8}\left(B+\frac{A}{2}\right)-6\right)$
Multiplying through by 16, $10B+5A+96=8A+6B+3A-96\Leftrightarrow 6A-4B=192$
Substituting, $6A-3A=192\Rightarrow A=\underline{\mathbf{64}},\ B=\underline{\mathbf{48}}$.

3. Since $(Rate)(Time)=Distance$, at $x$ mph, the 8 mile walk would take $\frac{8}{x}$ hours.
Thus, for total time *in hours*, we have $\frac{2}{x}+\frac{6}{x+0.25}=\frac{8}{x}-6\cdot\frac{1}{60}\Leftrightarrow\frac{6}{x+0.25}=\frac{6}{x}-\frac{1}{10}$
Multiplying through by $10x(x+0.25)$, $60x=60(x+0.25)-x(x+0.25)$
$\Leftrightarrow 0=15-x^2-\frac{x}{4}\Leftrightarrow 4x^2+x-60=(4x-15)(x+4)=0\Rightarrow x=\frac{15}{4}$

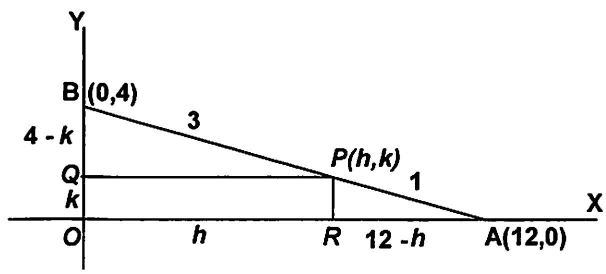
$\Rightarrow T=\frac{8}{15/4}=\frac{32}{15}$ hours $=2\frac{2}{15}$ hours = 2 hours 8 minutes $\Rightarrow \underline{(\mathbf{2,8})}$.

**Detailed Solutions for GBML Meet 3 - DECEMBER 2015**

**ROUND 2**

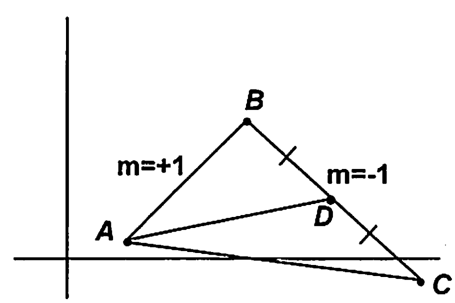
1. $L_1: 4x - y - 8 = 0 \Rightarrow P(2,0)$ and $Q(0,-8)$. Points equidistant from $P$ and $Q$ lie on the perpendicular bisector of $\overline{PQ}$. The midpoint of $\overline{PQ}$ is $M(1,-4)$ and the slope of $\overline{PQ}$ is $\frac{4}{1}$.
The required equation is $(y+4) = -\frac{1}{4}(x-1) \Leftrightarrow x + 4y + 15 = 0$ and $(A,B,C) = \underline{\mathbf{(1,4,15)}}$.

2. Since $A(3n,0)$, $B(0,n)$ and $C(6,2)$ are collinear, we have $\frac{n-0}{0-3n} = \frac{2-n}{6-0}$.
Cross multiplying, $6n = -6n + 3n^2 \Leftrightarrow 3n^2 - 12n = 3n(n-4) = 0 \Rightarrow n = 0, 4$. $n = 0$ is rejected, since this would imply that $A$ and $B$ were the same point $(0,0)$ and $AP : PB \neq 1:3$.
Thus, $n = 4 \Rightarrow A(12,0)$ and $B(0,4)$.
$\Delta APR \sim \Delta PBQ \Rightarrow \frac{1}{3} = \frac{12-h}{h} \Rightarrow h = 9$
$\Delta BQP \sim \Delta PRA \Rightarrow \frac{3}{1} = \frac{4-k}{k} \Rightarrow k = 1$
Thus, $P\underline{\mathbf{(9,1)}}$.




An alternate solution might use a weighted average:
$P = \frac{3A+B}{4} = \frac{3(12,0)+1(0,4)}{4} = \frac{(36,4)}{4} = \underline{\mathbf{(9,1)}}$.

3. Given: $A(3,1)$, $B(7,5)$, $C(13,-1)$
Since the slopes of $\overline{AB}$ and $\overline{BC}$ are $+1$ and $-1$ respectively, $\overline{AB} \perp \overline{BC}$ and $\overline{AB}$ is the altitude from vertex $A$.
$AB = \sqrt{(7-3)^2 + (5-1)^2} = \sqrt{16+16} = 4\sqrt{2}$.
If $\overline{AD}$ is the median from vertex $A$,
$D$ is the midpoint of $\overline{BC}$, $D(10,2)$ and
$AD = \sqrt{(10-3)^2 + (2-1)^2} = \sqrt{49+1} = 5\sqrt{2}$
Thus, $AB + AD = \underline{\mathbf{9\sqrt{2}}}$.



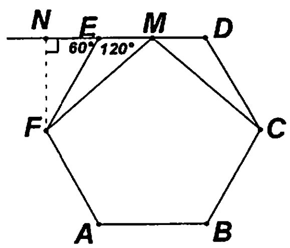
## Detailed Solutions for GBML Meet 3 - DECEMBER 2015

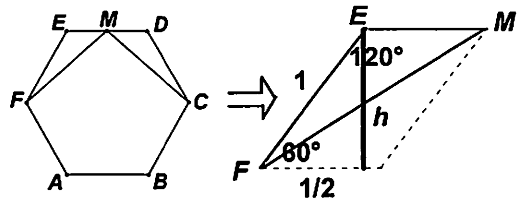
### ROUND 3

1. In any equilateral triangle, since the altitude ($h$) is opposite a 60° angle, the side $s = h \cdot \frac{2}{\sqrt{3}}$.

   $h = 4\sqrt{3} \Rightarrow s = 8$. $A(Eq.\Delta) = \frac{s^2\sqrt{3}}{4} = 16\sqrt{3}$. Since a rhombus is a parallelogram,

   $A(rhombus) = bh \Rightarrow \left(\frac{24\sqrt{3}}{4}\right)h = 16\sqrt{3} \Rightarrow h = \underline{\frac{8}{3}}$.



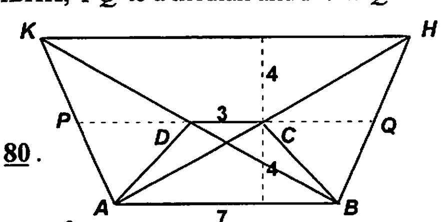




2. If a perpendicular is dropped from $F$ to $\overline{DE}$, $FN = \frac{1}{2}\sqrt{3}$ and the combined area of $\Delta FEM$ and $\Delta CDM$ is $2\left(\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$. Thus, the area of $ABCMF$ is $6\left(\frac{\sqrt{3}}{4}\cdot 1^2\right) - \frac{\sqrt{3}}{4} = \underline{\frac{5\sqrt{3}}{4}}$.

   Alternately, note that $\Delta FEM \cong \Delta CDM$ and a hexagon consists of 6 congruent equilateral triangles and has an area given by $A = \frac{3}{2}s^2\sqrt{3}$. Since one of the angles of $\Delta FEM$ measures 120°, the sum of the other two angles is 60°. So these two triangles can be rearranged to form a parallelogram with adjacent angles of 60° and 120° and sides of ½ and 1, giving an altitude of length $h = \frac{\sqrt{3}}{2}$ and the same result follows. Also, the area of a triangle can be computed as $\frac{1}{2}ab\sin C$, whenever two sides and the included angle are known and the same result follows.

3. Since $C$ and $D$ are midpoints of the diagonals of trapezoid $ABHK$, $\overline{PQ}$ is a median and $P$ and $Q$ are midpoints of $\overline{KA}$ and $\overline{HB}$ respectively. Since $\Delta KPD \sim \Delta KAB$, $PD = CQ = \frac{7}{2} \Rightarrow PQ = 10$.

   Thus, $PQ = \frac{KH + AB}{2} \Rightarrow KH = 13$ and $A = \frac{1}{2}\cdot 8\cdot(7+13) = \underline{80}$.




   Alternately, since the altitude and the sum of the bases of trapezoid $ABHK$ are twice the altitude and the sum of the bases of trapezoid $ABCD$ respectively, the area of $ABHK$ is 4 times the area of $ABCD$. Thus, $4\left(\frac{1}{2}\cdot 4\cdot(3+7)\right) = \underline{80}$.

**Detailed Solutions for GBML Meet 3 - DECEMBER 2015**

**ROUND 4**

1. $\left(\sqrt{3\cdot 5^{1/3}}\right)^6\cdot\sqrt[3]{3^{1/2}\cdot 5^{1/4}}=15^x\cdot 3^y$

$$\left(\left(3\cdot 5^{1/3}\right)^{1/2}\right)^6\cdot\left(3^{1/2}\cdot 5^{1/4}\right)^{1/3}=3^3 5^1 3^{1/6} 5^{1/12}=3^{19/6}\cdot 5^{13/12}$$

$$15^x\cdot 3^y=3^{x+y}\cdot 5^x\Rightarrow\begin{cases}x+y=19/6\\x=13/12\end{cases}\Rightarrow(x,y)=\underline{\left(\mathbf{\frac{13}{12},\frac{25}{12}}\right)}.$$

2. $\log_2\log_3(6x)=1-\log_a\left(\frac{1}{a}\right)\Leftrightarrow\log_2\log_3(6x)=1+1=2$

$$\log_3(6x)=2^2=4\Leftrightarrow 6x=3^4=81\Rightarrow x=\underline{\mathbf{\frac{27}{2}}}.$$

3. $x=(\log_2 5+\log_4 0.2)(\log_5 2+\log_{25}0.5)$

$$(\log_2 5+\log_4 0.2)=\log_4(25)+\log_4\left(\frac{1}{5}\right)=\log_4\left(25\cdot\frac{1}{5}\right)=\log_4 5$$

$$(\log_5 2+\log_{25}0.5)=\log_{25}4+\log_{25}0.5=\log_{25}(4\cdot 0.5)=\log_{25}2$$

$$x=\log_4 5\cdot\log_{25}2=\frac{1}{\log_5 4}\cdot\log_{25}2=\frac{1}{\log_{25}16}\cdot\log_{25}2=\frac{\log_{25}2}{\log_{25}16}=\log_{16}2=\underline{\mathbf{\frac{1}{4}}}.$$

**Detailed Solutions for GBML Meet 3 - DECEMBER 2015**

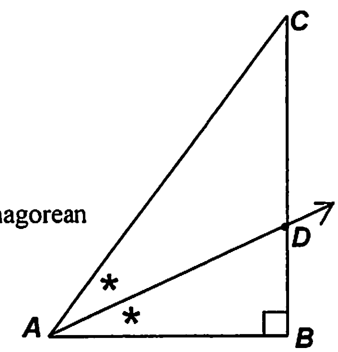
**ROUND 5**

1. $\tan x + \cot x = 4 \Leftrightarrow \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = 4 \Leftrightarrow \dfrac{\sin^2 x + \cos^2 x}{\sin x \cdot \cos x} = 4$

   $\Leftrightarrow \dfrac{1}{\sin x \cdot \cos x} = 4 \Rightarrow \sin x \cdot \cos x = \underline{\mathbf{\dfrac{1}{4}}}$ .

2. By the angle bisector theorem, $\dfrac{CD}{DB} = \dfrac{AC}{AB} \Rightarrow \dfrac{AC}{AB} = \dfrac{25}{7}$.
   Since $ABC$ is a right triangle, we recognize the 7-24-25 Pythagorean triple and $(AB, BC, AC) = (7a, 24a, 25a)$. Thus,

   $\tan C = \dfrac{7a}{24a} = \underline{\dfrac{7}{24}}$ .




3. $\sin 2x - \sin x + 1 = \cos 2x + \cos x$

   $\Leftrightarrow \sin 2x - \cos 2x = \sin x + \cos x - 1$

   $\Leftrightarrow 2\sin x \cos x - \left(1 - 2\sin^2 x\right) = \sin x + \cos x - 1$

   $\Leftrightarrow 2\sin x \cos x + 2\sin^2 x \cancel{-1} = \sin x + \cos x \cancel{-1}$

   $\Leftrightarrow 2\sin x\left(\cos x + \sin x\right) - \left(\sin x + \cos x\right) = 0$

   $\Leftrightarrow \left(2\sin x - 1\right)\left(\cos x + \sin x\right) = 0$

   $\Rightarrow \sin x = \dfrac{1}{2} \Rightarrow x = \underline{\mathbf{30°, 150°}}$ or $\sin x = -\cos x \Rightarrow \tan x = -1 \Rightarrow x = \underline{\mathbf{135°, 315°}}$.

**Detailed Solutions for GBML Meet 3 - DECEMBER 2015**

**TEAM ROUND**

1. We are given $\begin{cases}(1)\ 2S = 3F + 1K \\ (2)\ 3S + 2K + 1F = 25\end{cases}$

   Doubling (1), $2K = 4S - 6F$

   Substituting in (2), $3S + (4S - 6F) + 1F = 25 \Rightarrow 7S - 5F = 25 \Rightarrow \boxed{S = \frac{5(F+5)}{7}}$.

   Thus, 7 must be a factor of $(F+5)$ and the minimum value of $F$ is 2 and $(S,F,K) = \underline{(\mathbf{5,2,4})}$.

   Note: $F = 9 \Rightarrow S = 10 \Rightarrow K < 0$ and the given solution is unique.

2. $\log_2\left(\frac{2}{\sqrt{\cos x}}\right) = 2 - \frac{1}{2}\log_{\sec x}(2)$

   $\Leftrightarrow 1 - \frac{1}{2}\log_2(\cos x) = 2 - \frac{1}{2}\cdot\frac{1}{\log_2(\sec x)} = 2 - \frac{1}{2}\cdot\frac{1}{\log_2\left((\cos x)^{-1}\right)} = 2 + \frac{1}{2\log_2(\cos x)}$

   Let $Y = \log_2(\cos x)$. Then: $1 - \frac{1}{2}Y = 2 + \frac{1}{2Y} \Rightarrow 2Y - Y^2 = 4Y + 1 \Rightarrow Y^2 + 2Y + 1 = (Y+1)^2 = 0$

   $Y = \log_2(\cos x) = -1 \Leftrightarrow \cos x = 2^{-1} = \frac{1}{2} \Rightarrow \cos x = \underline{\mathbf{60°, 300°}}$.

3. Using the Pythagorean theorem on $\Delta PQR$, $z^2 + (x+y)^2 = 49$ (1).

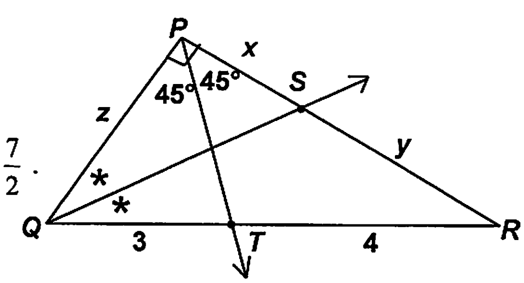
   Using the angle bisector theorem for bisector

   $\overline{PT}$: $\frac{z}{3} = \frac{x+y}{4} \Leftrightarrow x + y = \frac{4}{3}z$ (2) [Substituting for $x+y$ in (1), $\frac{25}{9}z^2 = 49 \Rightarrow z = \frac{21}{5}$.]

   $\overline{QS}$: $\frac{z}{x} = \frac{7}{y} \Leftrightarrow y = 7x\left(\frac{1}{z}\right) = 7x\left(\frac{5}{21}\right) = \frac{5}{3}x$ (3)

   $x + y = \frac{4}{3}\cdot\frac{21}{5} = \frac{28}{5}$

   $x + y = x + \frac{5}{3}x = \frac{28}{5} \Rightarrow x = \frac{28}{5}\cdot\frac{3}{8} = \frac{21}{10}$, $y = \frac{5}{3}\cdot\frac{21}{10} = \frac{7}{2}$.




   Applying Stewart's Theorem in $\Delta PQR$ for transversal $\overline{QS}$, we have

   $\left(\frac{21}{5}\right)^2\cdot\frac{7}{2} + 7^2\cdot\frac{21}{10} = QS^2\cdot\frac{28}{5} + \frac{21}{10}\cdot\frac{7}{2}\cdot\frac{28}{5} \Leftrightarrow$

   $\frac{49}{10}\left(\frac{63}{5} + 21\right) = \frac{28}{5}\left(QS^2 + \frac{147}{20}\right)$. Multiplying through by $\frac{5}{7}$, we have $\frac{7}{2}\left(\frac{168}{5}\right) = 4QS^2 + \frac{147}{5}$.

   $\Rightarrow QS^2 = \frac{1}{4}\left(\frac{7\cdot 84 - 147}{5}\right) = \frac{1}{4}\cdot\frac{588-147}{5} = \frac{1}{4}\cdot\frac{21^2}{5} \Rightarrow QS = \frac{21}{2\sqrt{5}} = \underline{\mathbf{\frac{21\sqrt{5}}{10}}}$.

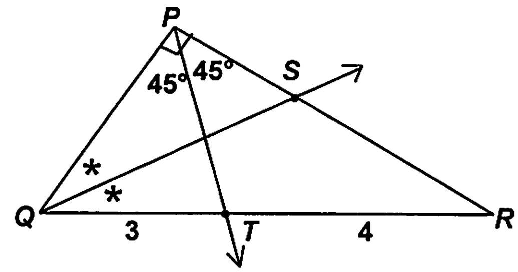
## Detailed Solutions for GBML Meet 3 - DECEMBER 2015

**TEAM ROUND - continued**

3. Alternate solution #1:

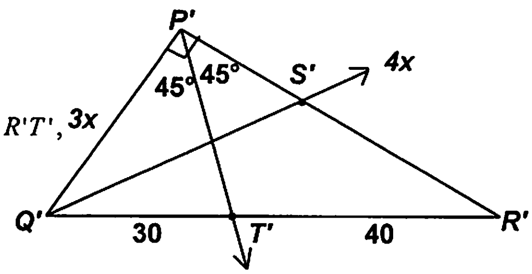
$$QS^2 = QR \cdot PQ - RS \cdot SP = 7 \cdot \frac{21}{5} - \frac{7}{2} \cdot \frac{21}{10}$$

$$= 147 \cdot \frac{3}{20} = \frac{3^2 \cdot 7^2}{2^2 \cdot 5} \Rightarrow QS = \frac{21}{2\sqrt{5}} = \underline{\frac{21\sqrt{5}}{10}}.$$




Alternate solution #2:
First multiply the given side lengths by 10.
By the angle bisector theorem, $Q'P' : R'P' = Q'T' : R'T'$,
so we let $Q'P' = 3x$ and $R'P' = 4x$.
Applying the Pythagorean Theorem to $\Delta P'Q'R'$,

$$(3x)^2 + (4x)^2 = 70^2 \Rightarrow x^2 = \frac{70^2}{25} \Rightarrow x = \frac{70}{5} = 14.$$




Thus, $Q'P' = 42$, $R'P' = 56$ and $\frac{R'S'}{P'S'} = \frac{70}{42} = \frac{5}{3}$, so we let
$P'S' = 3y$ and $R'S' = 5y \Rightarrow 8y = 56 \Rightarrow y = 7$.
Now right $\Delta P'Q'S'$ is similar to a right triangle with sides 1, 2, $\sqrt{5}$ and $Q'S' = 21\sqrt{5}$.
Dividing by 10 returns us to the original diagram, and

$$QS = \underline{\frac{21\sqrt{5}}{10}}.$$