GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2016

ROUND 1 - Algebra 1: Fractions and Word Problems

1. ____________ : ____________

2. ________________________

3. ________________________ feet

1. Let A be the sum of the values generated by the expression $(6n^2-13n+6)(4n+3)$ for $n=1,2,3,\ldots,10$.
   Let B be the sum of the values generated by the expression $(8n^2-6n-9)(3n-2)$ for $n=1,2,3,\ldots,10$.
   Compute the ratio $A:B$.

2. Let A and B be the two smallest prime numbers which are the sum of 5 distinct prime numbers.
   Compute $\dfrac{|A-B|}{A+B}$.
   Remember: 1 is <u>not</u> a prime number.

3. At a New Year's Eve celebration, Marty set a fuse for his vertical cannon so that the blast would occur in exactly 26 seconds. If he then immediately ran directly away in a straight line path at 27 feet per second, how far, to the nearest foot, had he run when he heard the blast?
   Note: Assume sound travels at a rate of 1080 feet per second.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2016

ROUND 2 - Coordinate Geometry of the Straight Line

1. ______________________

2. ______________________

3. (__________ , __________)

1. Lines L_1 and L_2 are two parallel lines which together cut off segments of 3 inches on the x-axis and 4 inches on the y-axis. The perpendicular distance (in inches) between L_1 and L_2 is k. Compute k.

2. The following four lines are tangent to circle O:
$$\begin{cases} x+2y=a \\ x+2y=b \\ x+y=11 \\ x+y=17 \end{cases}$$
If $a \neq b$, compute $|a-b|$.

3. The location of point $P(x, y)$ is changed as follows (in the specified order):
    - reflected in the line $x = 0$ and then
    - reflected in the line $y = 1$ and then
    - reflected across the line $y = x$, and finally
    - reflected through the origin.

   In terms of x and y, what are the coordinates of the final image point?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2016

ROUND 3 - Geometry - Polygons: Area and Perimeter

1. ______________________

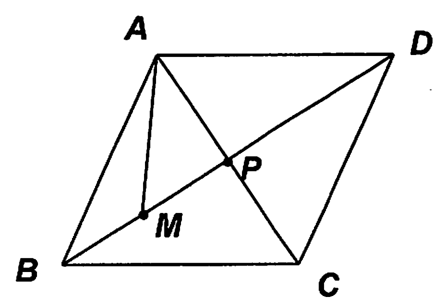
2. ______________________sq. units

3. ______________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. The diagonal of a square and the altitude of an equilateral triangle have equal lengths. The ratio of the area of the square to the equilateral triangle is $K : 1$. Compute K.

2. The area of rhombus $ABCD$ is 240 square units. If $BD = 7k + 2$, $AC = 4k$, and $DM : MB = 7:3$, compute the area of ΔAPM .



3. The lengths of the sides of a triangle are 2, 3, and x. Compute x, if the area of the triangle is also x.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2016

ROUND 4 - Algebra 2: Logs, Exponents Radicals and equations involving them

1. ______________________

2. ______________________

3. ______________________

1. Compute, in radians, the <u>least</u> angle x in the interval $5\pi < x < 6\pi$ for which $\log_2(\cos x) = -\frac{1}{2}$.

2. If $b = \log_3 x$, compute the real value of x which satisfies $\log_b\left(\log_3 x^2\right) = 2$.

3. Given: $\log_{\sqrt{3}}\left(9\sqrt{3}\right) = t$ and $\log_{20} 2 = k$

   Find a simplified expression for $\log_{20}\left(\frac{t}{2}\right)$ in terms of k.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2016

ROUND 5 - Trig Analysis and Complex Numbers in Trig Form

1. (____________, ____________)

2. ________________________

3. (____________, ____________)

1. For a minimum positive integer n, $\left(\sqrt{2}cis\left(9°\right)\right)^n = A$, a real number. Compute the ordered pair $\left(n, A\right)$.

2. Find <u>all</u> values of x over $0° \le x < 360°$ for which

$$\sin 145° = \cos 215° \cdot \tan\left(270° - x\right) \cdot \sec\left(-540°\right)$$

3. In ΔABC, $\sin A : \sin B : \sin C = 4:5:6$ and $\cos A : \cos B : \cos C = x : y : 2$. Compute the ordered pair of <u>real</u> numbers $\left(x, y\right)$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2016

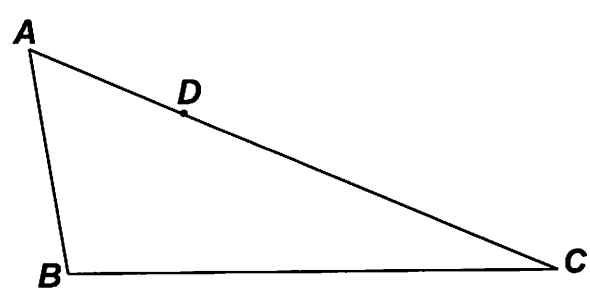
TEAM ROUND

3 pts.	1. ____________________	sq. units
3 pts.	2. (__________ , __________)	
4 pts.	3. ____________________	sq. units

1. Given: $f=\{(3,4),\ (6,0)\}$
   Compute the area of the closed region formed by joining the points of the functions f and f^{-1}.

2. In simplified form, $\left(30-12\sqrt{6}\right)^{\frac{1}{2}}\left(\frac{\sqrt{2}}{\sqrt{3}}-\frac{\sqrt{3}}{\sqrt{2}}\right)=\sqrt{A}-\sqrt{B}$, compute the ordered pair (A,B).

3. In ΔABC, $AC=18$ and D is the point on $\overline{AC}$ for which $AD=5$. Perpendiculars drawn from D to $\overline{AB}$ and $\overline{CB}$ have lengths of 4 and 5 respectively. Compute the area of ΔABC.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2016

Answer Sheet

Round 1

1. 1 : 1
2. $\frac{2}{45}$
3. 720

Round 2

1. $\frac{12}{5}$
2. $3\sqrt{10}$
3. $(y-2, x)$

Round 3

1. $\frac{\sqrt{3}}{2}$
2. 24
3. $\sqrt{5}$

Round 4

1. $\frac{23\pi}{4}$
2. 9
3. $1 - 3k$

Round 5

1. (20, -1024)
2. 55°, 235°
3. $(12,9)$

Team Round

1. $\frac{7}{2}$ or 3.5 (3 pts)
2. $(2,3)$ (3 pts)
3. $\frac{360}{7}$ (4 pts)

Detailed Solutions for GBML Meet 3 - DECEMBER 2016

ROUND 1

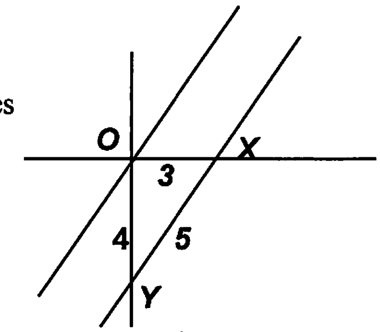
1. After evaluating A and B for $n = 1, 2$ (-7 and 11) or, after confirming the complete factorization of A and $B\left[A=(3n-2)(2n-3)(4n+3) \text{ and } B=(4x+3)(2x-3)(3x-2)\right]$, we realize the required ratio is **<u>1 : 1</u>**.

2. The smallest 10 primes are 2, **3**, **5**, 7, **11**, **13**, 17, 19, 23, and 29.
If one of the 5 chosen primes were even (i.e., 2), then the other four would be odd, and the resulting sum would be even; hence not prime. Thus, the sum must be comprised of 5 odd primes.
The sum of the 5 smallest primes is 39. Swapping out any of the above bolded primes increases the sum by at least 4. Thus, 41 is not possible.
The target primes are 43, 47, 53, etc.
Interchanging 17 for 13 increases the sum by $4 \Rightarrow A = 43$.
Interchanging 11 for 19 increases the sum by $8 \Rightarrow B = 47$.
Thus, $\dfrac{|A-B|}{A+B}=\dfrac{47-43}{47+43}=\dfrac{4}{90}=\underline{\mathbf{\dfrac{2}{45}}}$.

3. Let t denote the time (in seconds) it takes sound to reach Marty's ear.
Thus, Marty runs $1080t$ feet.
$1080t = 27(26) + 27t \Rightarrow 1053t = 702$
and $1080t=\dfrac{1080(702)}{1053}=\dfrac{2^3\cdot 3^3\cdot 5\left(2\cdot 3^3\cdot 13\right)}{3^4\cdot 13}=2^4\cdot 3^2\cdot 5=9(80)=\underline{\mathbf{720}}$.

Detailed Solutions for GBML Meet 3 - DECEMBER 2016

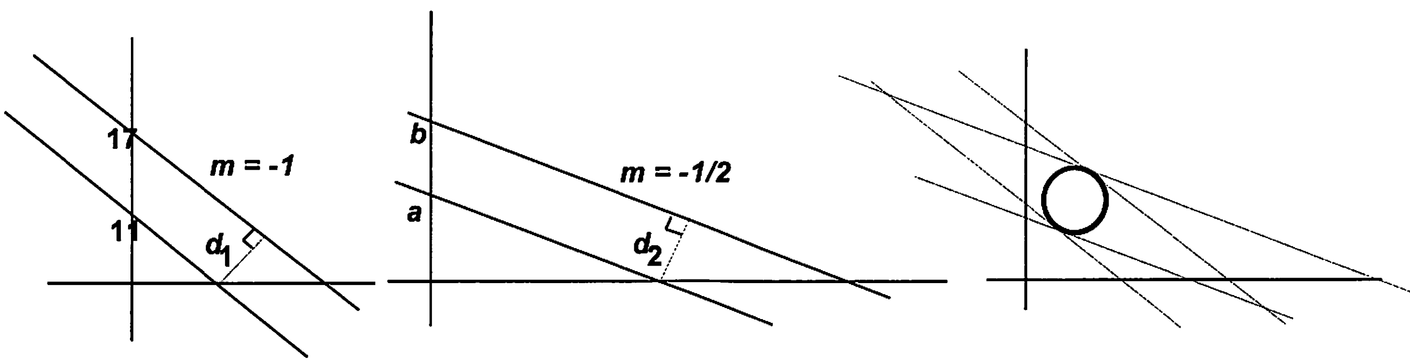
ROUND 2



1. Assume (with no loss of generality) that one of the parallel lines passes through the origin and has a positive slope. The distance between the lines is the altitude to the hypotenuse of a 3-4-5- right triangle.
   $A(\Delta OXY)=\frac{1}{2}\cdot 3\cdot 4=\frac{1}{2}\cdot 5\cdot h \Rightarrow h=\underline{\mathbf{\frac{12}{5}}}$.

$$\begin{cases} x+2y=a \\ x+2y=b \\ x+y=11 \\ x+y=17 \end{cases}$$

2. We require that the distance between the pairs of parallel lines be the same.
   Using the point-to-line distance formula, $\frac{|a-b|}{\sqrt{5}}=\frac{|-6|}{\sqrt{2}}=3\sqrt{2}\Rightarrow |a-b|=\underline{\mathbf{3\sqrt{10}}}$.



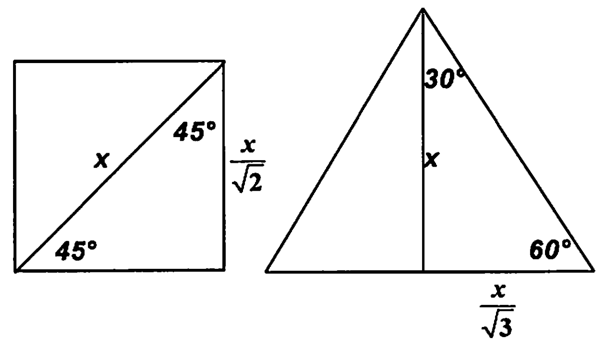
3. Reflecting across $x=0$ (the y-axis), replaces the x-coordinate with its opposite $(x,y)\rightarrow(-x,y)$.
   Reflecting across the horizontal line $y=1$:
   The distance from any point $P(a,b)$ to $y=1$ is $|b-1|$.
   If $b>1$, then P is above $y=1$, $|b-1|=b-1$, and the image P' is $b-1$ units below $y=1$, namely $P'(a,1-(b-1))=(a,2-b)$.
   If $b<1$, then P is below $y=1$, $|b-1|=1-b$ and the image P' is $1-b$ units above $y=1$, namely $P'(a,1+(1-b))=(a,2-b)$. If $b=1$, then $P=P'$ and again we have $P'(a,2-b)$
   Thus, $(x,y)\rightarrow(-x,y)\rightarrow\underline{(-x,2-y)}$
   Reflecting across $y=x$, swaps the x- and y-coordinates. $\rightarrow(2-y,-x)$
   Reflecting through the origin, simply swaps the signs of the x- and y-coordinates. $\rightarrow\underline{(y-2,x)}$
   Ex: $(3,4)\rightarrow(-3,4)\rightarrow(-3,-2)\rightarrow(-2,-3)\rightarrow(2,3)$
   $(-2,-5)\rightarrow(2,-5)\rightarrow(2,7)\rightarrow(7,2)\rightarrow(-7,-2)$

Detailed Solutions for GBML Meet 3 - DECEMBER 2016

ROUND 3

1. The area of the square = $\frac{x^2}{2}$. The area of the equilateral triangle is $\frac{1}{2}\cdot\frac{2x}{\sqrt{3}}\cdot x=\frac{x^2}{\sqrt{3}}$.

   The required ratio is $\frac{\sqrt{3}}{2}:1 \Rightarrow K=$ $\underline{\frac{\sqrt{3}}{2}}$.



2. Since the area of a rhombus is given by $\frac{1}{2}d_1\cdot d_2$, we have

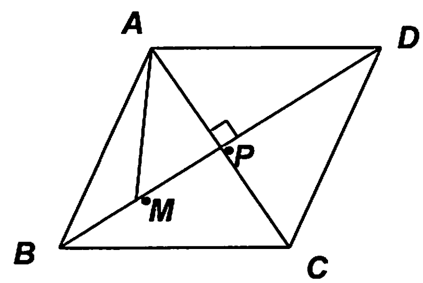
   $\frac{1}{2}(7k+2)(4k)=240 \Leftrightarrow (7k+2)k=120$

   $\Leftrightarrow 7k^2+2k-120=(7k+30)(k-4)=0 \Rightarrow k=4$

   $\Rightarrow AC=16,\ BD=30 \Rightarrow AP=8,\ BP=PD=15$.

   Let $MP=x$. Then: $\frac{15+x}{15-x}=\frac{7}{3} \Rightarrow 45+3x=105-7x \Rightarrow x=6$

   Thus, the area of ΔAPM is $\frac{1}{2}\cdot 6\cdot 8=\underline{\mathbf{24}}$.



3. The semi-perimeter is $\frac{x+5}{2}$. Using Heron's formula,

   $\text{Area}=\sqrt{\frac{x+5}{2}\cdot\frac{x+1}{2}\cdot\frac{x-1}{2}\cdot\frac{5-x}{2}}=\frac{1}{4}\sqrt{(25-x^2)(x^2-1)}=x$

   $\Rightarrow (25-x^2)(x^2-1)=16x^2 \Leftrightarrow x^4-10x^2+25=(x^2-5)^2=0$

   $\Rightarrow x=\underline{\sqrt{5}}$.

Detailed Solutions for GBML Meet 3 - DECEMBER 2016

ROUND 4

1. $\log_2(\cos x) = -\frac{1}{2} \Rightarrow \cos x = 2^{-\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow x = \pm\frac{\pi}{4} + n(2\pi) \Rightarrow 6\pi - \frac{\pi}{4} = \underline{\mathbf{\frac{23\pi}{4}}}$.

2. $\log_b(\log_3 x^2) = 2 \Leftrightarrow \log_3 x^2 = b^2 \Leftrightarrow 2\log_3 x = b^2$

   Substituting and transposing terms, $(\log_3 x)^2 - 2\log_3 x = 0$.

   $\Leftrightarrow \log_3 x(\log_3 x - 2) = 0$.

   If $\log_3 x = 0$, then $x = 3^0 = 1$. However, $x = 1$ and $b = \log_3 x \Rightarrow b = 0$, which must be rejected, since the base of a logarithm cannot be zero. Thus, $x = 1$ is rejected.

   Since, for all *x*, $\log_3 x > 0$, $\log_3 x - 2 = 0 \Rightarrow x = 3^2 = \underline{\mathbf{9}}$.

3. $t = \log_{\sqrt{3}}(9\sqrt{3}) = \log_{\sqrt{3}} 9 + \log_{\sqrt{3}}\sqrt{3} = \log_{\sqrt{3}}(\sqrt{3})^4 + \log_{\sqrt{3}}\sqrt{3} = 4 + 1 = 5$.

   $\log_{20}\left(\frac{t}{2}\right) = \log_{20}\left(\frac{5}{2}\right) = \log_{20} 5 - \log_{20} 2 = \log_{20}\left(\frac{20}{4}\right) - \log_{20} 2 = 1 - 2\log_{20} 2 - \log_{20} 2 = \underline{\mathbf{1 - 3k}}$.

Detailed Solutions for GBML Meet 3 - DECEMBER 2016

ROUND 5

1. $\left(\sqrt{2}cis(9°)\right)^n = \left(\sqrt{2}\right)^n\left(\cos(9n)° + i\sin(9n)°\right)$. If this is to evaluate to a real number, we require that $\sin(9n)° = 0$, for a minimum positive value of *n*.

   Thus, $n = 20 \Rightarrow A = \left(\sqrt{2}\right)^{20}\cos(180°) = 2^{10}(-1) = -1024$ and $(n, A) = \underline{\mathbf{(20,-1024)}}$.

2. Using reduction formulas, $\sin 145° = \cos 215° \cdot \tan(270° - x) \cdot \sec(-540°)$
   $\Leftrightarrow \sin 35° = -\cos 35° \cdot \cot x \cdot \sec(180°)$
   $\Leftrightarrow \tan 35° = -\cot x \cdot (-1) = \cot x = \tan(90° - x)$
   $\Leftrightarrow 90° - x = 35° \pm n \cdot 180°$, for integer values of *n*, since the function defined by $y = \tan x$ is periodic, with a period of $180°$.
   $\Leftrightarrow x = 55° \pm n \cdot 180°$
   $\Rightarrow x = \underline{\mathbf{55°, 235°}}$

3. According to the Law of Sines, the sides of ΔABC must be multiples of 4, 5, and 6, so call them $a = 4n$, $b = 5n$, and $c = 6n$. Applying the law of Cosines from each angle's perspective, we have

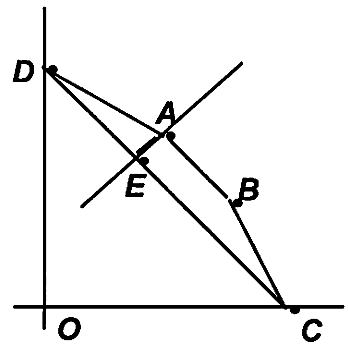
$$\begin{cases} 16n^2 = 61n^2 - 60n^2\cos A \\ 25n^2 = 52n^2 - 48n^2\cos B \\ 36n^2 = 41n^2 - 40n^2\cos C \end{cases} \Rightarrow \begin{cases} \cos A = \dfrac{45}{60} = \dfrac{3}{4} \\ \cos B = \dfrac{27}{48} = \dfrac{9}{16} \\ \cos C = \dfrac{5}{40} = \dfrac{1}{8} \end{cases}$$

   Thus, $\cos A : \cos B : \cos C = \frac{3}{4} : \frac{9}{16} : \frac{1}{8} = 12 : 9 : 2 \Rightarrow (x, y) = \underline{\mathbf{(12,9)}}$.

Detailed Solutions for GBML Meet 3 - DECEMBER 2016

TEAM ROUND

1. Since $f=\{(3,4),(6,0)\}$, $f^{-1}=\{(4,3),(0,6)\}$.
The required quadrilateral region is bounded by trapezoid *ABCD*.
Since the coordinates of *A*, *B*, *C*, and *D* are known, applying the point-to-point distance formula gives us $AB=\sqrt{2}$ and $CD=6\sqrt{2}$.

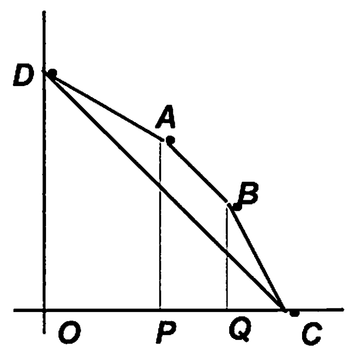


$$\begin{cases}\overline{DC}: x+y=6\ (m=-1)\\ \overline{AE}: (y-4)=+1(x-3)\end{cases}\Rightarrow x=\frac{5}{2}\Rightarrow E\left(\frac{5}{2},\frac{7}{2}\right)$$

$$AE=\sqrt{\left(3-\frac{5}{2}\right)^2+\left(4-\frac{7}{2}\right)^2}=\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}$$

Thus, the area of *ABCD* is $\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\left(\sqrt{2}+6\sqrt{2}\right)=\frac{\sqrt{2}}{4}\cdot 7\sqrt{2}=\underline{\frac{7}{2}}$.

Solution #2:
If you didn't realize *ABCD* was a trapezoid, the pentagonal region *OCBAD* could be subdivided into two trapezoids and a triangle. The required area can be obtained by subtracting out the area of ΔDOC.
Let | | denote "area of…" .



$$|BQC|+|APQB|+|DOPA|-|DOC|$$
$$=\frac{1}{2}\cdot 2\cdot 3+\frac{1}{2}\cdot 1\cdot(3+4)+\frac{1}{2}\cdot 3\cdot(4+6)-\frac{1}{2}\cdot 6\cdot 6=3+\frac{7}{2}+15-18=\underline{\frac{7}{2}}\ .$$

2. $\left(30-12\sqrt{6}\right)^{\frac{1}{2}}=\sqrt{30-12\sqrt{6}}=\sqrt{6\left(5-2\sqrt{6}\right)}=\sqrt{6}\sqrt{5-2\sqrt{6}}$

$$\frac{\sqrt{2}}{\sqrt{3}}-\frac{\sqrt{3}}{\sqrt{2}}=\frac{\sqrt{6}}{3}-\frac{\sqrt{6}}{2}=-\frac{\sqrt{6}}{6}$$

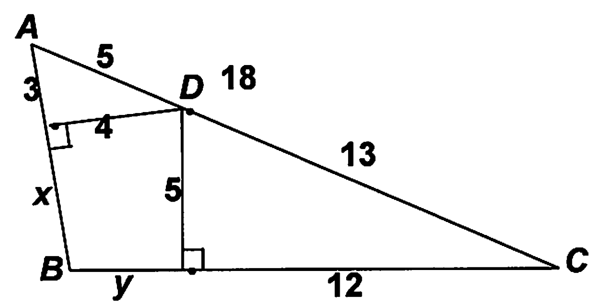
$$\therefore\left(30-12\sqrt{6}\right)^{\frac{1}{2}}\left(\frac{\sqrt{2}}{\sqrt{3}}-\frac{\sqrt{3}}{\sqrt{2}}\right)=\sqrt{6}\cdot\left(-\frac{\sqrt{6}}{6}\right)\sqrt{5-2\sqrt{6}}=-\sqrt{5-2\sqrt{6}}$$

If this can be simplified further, then $5-2\sqrt{6}$ must be expressible as a perfect square of a NEGATVE number.

$$5-2\sqrt{6}=(A+B)^2\Rightarrow\begin{cases}A^2+B^2=5\\ AB=-\sqrt{6}\end{cases}$$

Clearly, *A* and *B* have opposite signs and $\pm\sqrt{2}, \mp\sqrt{3}$ satisfy these equations.
Since the answer must be negative, we have $\sqrt{2}-\sqrt{3}\Rightarrow\underline{(\mathbf{2,3})}$.

TEAM ROUND - continued



3. $\sin B = \sin(180 - (A + C)) = \sin(A + C)$
$= \sin A \cos C + \sin C \cos A$
$= \frac{4}{5} \cdot \frac{12}{13} + \frac{5}{13} \cdot \frac{3}{5} = \frac{63}{65}$

The area of ΔABC is given by

$\frac{1}{2} 18(y+12)\sin C = \boxed{9(y+12) \cdot \frac{5}{13}}$ and by

$\frac{1}{2}(x+3)(y+12)\sin B = \boxed{\frac{1}{2}(x+3)(y+12) \cdot \frac{63}{65}}$

Equating and cancelling the common factors of 9, 13, and $(y+12)$,

$\frac{7(x+3)}{2 \cdot 5} = \frac{5}{1} \Rightarrow x = \frac{50}{7} - 3 = \frac{29}{7}$

Since $y^2 + 5^2 = BD^2 = x^2 + 4^2$, $y^2 = x^2 - 9 \Rightarrow y^2 = \frac{29^2 - 9(7^2)}{7^2} = \frac{400}{7^2} \Rightarrow y = \frac{20}{7}$.

Thus, the required area is $9\left(\frac{20}{7} + 12\right)\frac{5}{13} = 9\left(\frac{104}{7}\right)\left(\frac{5}{13}\right) = \frac{9 \cdot 8 \cdot \cancel{13} \cdot 5}{7 \cdot \cancel{13}} = \underline{\frac{\mathbf{360}}{\mathbf{7}}}$.