GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2017

ROUND 1 - Algebra 1: Fractions and Word Problems

1. ____________ : ____________

2. (____________ , ____________)

3. (____________ , ____________)

1. Given: $yz : zx : xy = 1:2:3$,
   express the ratio $\frac{y}{zx} : \frac{x}{yz}$ in simplest terms.

2. Two integers x and y have the following property:
   If the first number (x) is increased by 1 and the second number (y) is diminished by 3, the product is increased by 33. However, if the first number is diminished by 2 and the second is increased by 1, the product is increased by 21. Compute the ordered pair (x, y).

3. Travelling <u>from</u> point P <u>to</u> point Q, a distance of 48 miles, 32 miles are uphill and 16 miles are downhill. Nancy takes 1 hour and 44 minutes from P to Q, but returns from Q to P in only 88 minutes. Assume Nancy's uphill rate is j mph and her downhill rate is k mph and neither rate changes. Compute the ordered pair (j, k).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2017

ROUND 2 - Coordinate Geometry of the Straight Line

1. ______________________

2. ________ : ________ : ________

3. (____________ , ____________)

1. The distance from the point $P(k,-5)$ to the line $L_1 : \{(x,y) \mid 3x-4y-10=0\}$ is *k*. Compute all possible values of *k*.

2. Points *A* and *B* lie on segment $\overline{PQ}$. Furthermore, it is known that $\begin{cases} PA:AQ=3:11 \\ PB:BQ=13:8 \end{cases}$. Compute the ratio $PA:AB:BQ$.

3. Given: Points $T(-9,-4)$, $E(31,14)$, $K(20,17)$. $C(a,b)$ is the centroid of ΔTEK. Compute the ordered pair (a,b).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2017

ROUND 3 - Geometry - Polygons: Area and Perimeter

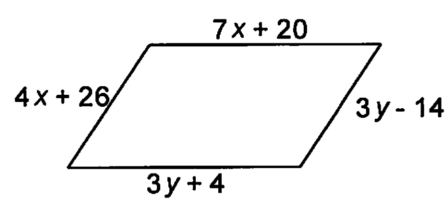
1. ______________________

2. ______________________

3. ______________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. Given the parallelogram *ABCD*. Compute $|y-x|$.



2. Let s be the length of a side of an equilateral triangle *ABC*, and x be the length of the side of a rhombus *PQRS*. If $m\angle PQR = m\angle A$ and the perimeter of the rhombus is the same as the perimeter of the equilateral triangle, the area of the rhombus, as a simplified expression, is ks^2. Compute k.

3. In rectangle *ABCD*, $AB = 6$, $AD = 30$ and G is the midpoint of $\overline{AD}$. $\overline{AB}$ is extended 2 units beyond B to point E, and F is the intersection of $\overline{ED}$ and $\overline{BC}$. Compute the area of *BFDG*.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2017

ROUND 4 - Algebra 2: Logs, Exponents Radicals and equations involving them

1. ______________________

2. $y =$ ______________________

3. ______________________

1. Given: $\log_{27} 81 - \log_3\left(\frac{x}{3}\right) + \log_{\sqrt{2}} 2 = 4$.

   Compute <u>all</u> values of x for which this is true.
   If necessary, express in terms of a simplified radical.

2. Given: $16(\log x)^2 + 9(\log y)^2 = 24(\log x)(\log y)$.
   Determine a simplified expression (without radicals) for y in terms of x.

3. Given: $\log_5 16 = a$
   Compute $\log_8 25$ in terms of a.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2017

ROUND 5 - Trig Analysis and Complex Numbers in Trig Form

1. ______________________

2. ______________________

3. ______________________

1. Let $S = i + 2i^3 + 4i^5 + 6i^7$.
   Express S in $rcis\theta$ form, where $0° \le \theta < 360°$ and $r > 0$.

2. Given: $\sin A = \frac{8}{9}$, $\cot A < 0$, $\tan B = \frac{1}{4}$, $\sin B < 0$.
   Compute $(\tan A)(\sin B) - (\sec A)(\cos B) + \cos^2 B$

3. Compute <u>all</u> values of x over $0° \le x < 360°$ for which
   $\sin(x + 30°) + \sin(x - 30°) = \cos(x + 60°) - \cos(x - 60°) - \tan 240°$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2017

TEAM ROUND

3 pts.	1.	______________
3 pts.	2.	______________
4 pts.	3.	______________

1. Given: $L_1:\{(x,y)\mid 3x-4y=4\}$ and $L_2:\{(x,y)\mid 4x-3y=10\}$,
   $L_3:\{(x,y)\mid x+y=-1\}$ and $L_4:\{(x,y)\mid x-y=3\}$,
   $L_1\cap L_2=P,\ L_1\cap L_3=A,\ L_1\cap L_4=B,\ L_3\cap L_4=C$.
   Compute $PA+PB+PC$.

2. Determine all values of θ over $0\le\theta<2\pi$ for which $\sin 2\theta\le\sin\theta$.

3. In a square *ABCD*, two equilateral triangles are constructed, one with base $\overline{AB}$ and the other with base $\overline{CD}$. The third vertex of each of the equilateral triangles is in the interior of the square. The intersection of the interiors of the two equilateral triangles is a region bounded by a rhombus. If the perimeter of the square is $8\sqrt{3}$, compute the positive difference between the lengths of the diagonals of this rhombus.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2017

Answer Sheet

Round 1

1. $1:4$
2. $(-19,-21)$
3. $(24,40)$

Round 2

1. 5
2. $9:17:16$
3. $(14, 9)$

Round 3

1. 16
2. $k=\frac{9\sqrt{3}}{32}$
3. 67.5

Round 4

1. $\sqrt[3]{3}$
2. $y=x^{\frac{4}{3}}$ (NOT $x\sqrt[3]{x}$)
3. $\frac{8}{3a}$

Round 5

1. $3cis(270°)$
2. $-\frac{12}{17}$
3. 210°, 330°

Team Round

1. 15 (3 pts)
2. 0 (3 pts)
   $\frac{\pi}{3}\le\theta\le\pi$
   $\frac{5\pi}{3}\le\theta<2\pi$
3. $8-4\sqrt{3}$ (4 pts)

Detailed Solutions for GBML Meet 3 - DECEMBER 2017

ROUND 1

1. Clearly, all the variables are nonzero.

   The required ratio $\frac{y}{zx}:\frac{x}{yz} \Leftrightarrow \frac{y}{zx}\cdot\frac{yz}{x}=\frac{y^2}{x^2}$.

   The given extended ratio $\underline{yz:zx}:xy=\underline{1:2}:3 \Rightarrow \frac{y\cancel{z}}{\cancel{z}x}=\frac{1}{2} \Rightarrow x=2y$

   Substituting, $\frac{y^2}{x^2}=\frac{y^2}{(2y)^2}=\frac{1}{4} \Rightarrow$ **<u>1:4</u>**.

2. Given: $\begin{cases}(x+1)(y-3)=xy+33\\(x-2)(y+1)=xy+21\end{cases}$

Multiplying out the product of the binomials (FOILing), we have

$$(x+1)(y-3)=\cancel{xy}-3x+y-3=\cancel{xy}+33 \Leftrightarrow y=3x+36$$
$$(x-2)(y+1)=\cancel{xy}+x-2y-2=\cancel{xy}+21 \Leftrightarrow x-2y=23$$

Substituting for y, $x-2(3x+36)=23 \Rightarrow -5x=95 \Rightarrow x=-19,\ y=3(-19)+36=-21$,

Thus, $(x,y)=\underline{\mathbf{(-19,-21)}}$.

3. Let $(r_U,r_D)=(x,y)$ denote Nancy's uphill and downhill rates (in miles per <u>minute</u>), respectively. Then:

   Travelling from P to Q: $\frac{32}{x}+\frac{16}{y}=104$

   Travelling from Q back to P, uphill and downhill reverse. Thus, $\frac{16}{x}+\frac{32}{y}=88$.

   Doubling the second equation and subtracting the first,

   $\frac{64-16}{y}=88\cdot2-104 \Leftrightarrow \frac{48}{y}=72 \Rightarrow y=\frac{2}{3}$ miles/min

   Substituting in the first equation for y, $\frac{32}{x}+\frac{16}{\frac{2}{3}}=104 \Leftrightarrow \frac{32}{x}=104-24=80 \Rightarrow x=\frac{2}{5}$.

   Converting from miles/minute to mph, we have $(x,y)=\underline{\mathbf{(24,40)}}$.

Detailed Solutions for GBML Meet 3 - DECEMBER 2017

ROUND 2

1. Using the point to line distance formula, we have

$$\frac{|3k-4(-5)-10|}{\sqrt{3^2+(-4)^2}}=k \Leftrightarrow |3k+10|=5k$$

Note that k cannot be negative!

$3k+10=5k \Leftrightarrow k=\underline{5}$

$3k+10=-5k \Leftrightarrow k=-\frac{5}{4}$ which is extraneous.

2. $PQ=PA+AQ \Leftrightarrow PQ=3a+11a=14a$, $PQ=PB+BQ \Leftrightarrow PQ=13b+8b=21b$

By transitivity, $\boxed{2a=3b}$.

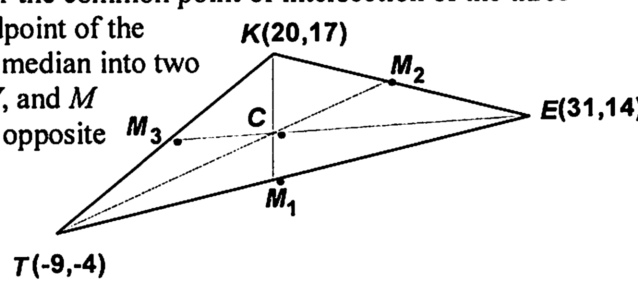
$PA:AB:BQ=3a:(13b-3a):8b$

$\Leftrightarrow 3a:\left(13\cdot\frac{2}{3}a-3a\right):\left(8\cdot\frac{2}{3}a\right) \Leftrightarrow 3a:\frac{17}{3}a:\frac{16}{3}a$

Cancelling the common factor and multiplying through by 3, we have $\underline{\mathbf{9:17:16}}$.

Alternately, PQ must be a multiple of both 3+11 and 13+8, so make it 42; the rest follows quickly.

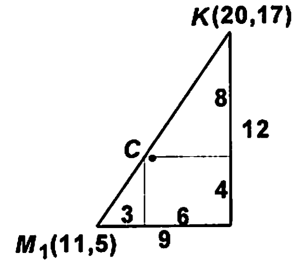
3. Recall: The centroid is the point of concurrency or the common point of intersection of the three medians (segments connecting a vertex to the midpoint of the opposite side). The centroid always divides <u>each</u> median into two segments whose lengths are in a 2:1 ratio. If C, V, and M denote the centroid, a vertex, and midpoint of the opposite side, then $VC:CM=2:1$.



The diagram at the right accurately represents the relative positions of the points (left, right, above and below), but the vertices were not plotted on a grid. So, for example, we should not assume that KM_1 is vertical.

Computing M_1 avoids fractions.

$M_1\left(\frac{-9+31}{2},\frac{-4+14}{2}\right)=(11,5)$ and we must determine the coordinates of the trisection point closer to the midpoint M_1.



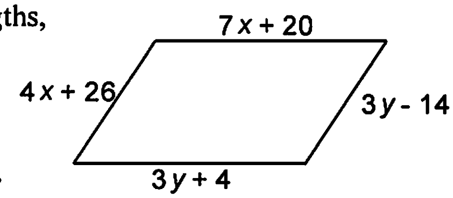
Thus, the centroid is 8 units below, and 6 units to the left of K, or 3 units to the right of, and 4 units above M_1.

In either case, the coordinates of C are **<u>(14, 9)</u>**.

Detailed Solutions for GBML Meet 3 - DECEMBER 2017

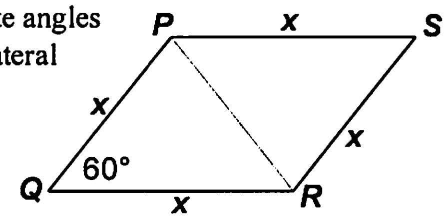
ROUND 3

1. Since opposite sides of a parallelogram must have equal lengths, we have
$4x+26=3y-14 \Leftrightarrow 4x-3y=-40$ and
$7x+20=3y+4 \Leftrightarrow 7x-3y=-16$
Subtracting, $3x=24 \Rightarrow x=8, y=24 \Rightarrow |y-x|=24-8=\underline{\mathbf{16}}$.

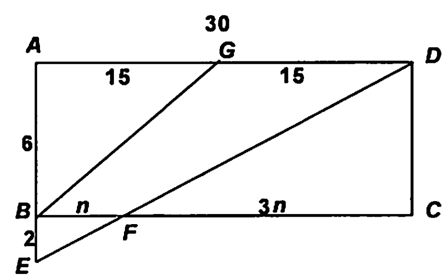


2. Since all angles of an equilateral triangle are 60°, the opposite angles of the rhombus must be 60° and 120°. The area of the equilateral triangle is $\frac{s^2\sqrt{3}}{4}$ and the area of the rhombus is $2\left(\frac{x^2\sqrt{3}}{4}\right)$
Equating perimeters, $3s=4x \Rightarrow x=\frac{3}{4}s$

$$2\left(\frac{x^2\sqrt{3}}{4}\right)=\frac{\left(\frac{3}{4}s\right)^2\sqrt{3}}{2}=\frac{9\sqrt{3}}{32}s^2 \Rightarrow k=\underline{\frac{9\sqrt{3}}{32}}.$$



3. $\Delta BEF \sim \Delta CDF \Rightarrow \frac{BF}{CF}=\frac{2}{6}=\frac{1}{3}$
$\Rightarrow 4n=30 \Rightarrow n=7.5 \Rightarrow CF=22.5$
Thus, the area of $BFDG$ is
$6\cdot 30-\frac{1}{2}\cdot 6\cdot 15-\frac{1}{2}\cdot 6\cdot 22.5=180-3(37.5)=\underline{\mathbf{67.5}}$.



Detailed Solutions for GBML Meet 3 - DECEMBER 2017

ROUND 4

1. $\log_{27} 81 - \log_3\left(\frac{x}{3}\right) + \log_{\sqrt{2}} 2 = 4 \Leftrightarrow \frac{4}{3} - \log_3 x + 1 + 2 = 4$

$\Rightarrow \log_3 x = 4\frac{1}{3} - 4 = \frac{1}{3}$

$\Rightarrow x = 3^{\frac{1}{3}} = \underline{\sqrt[3]{3}}$.

2. $16(\log x)^2 + 9(\log y)^2 = 24(\log x)(\log y)$

$\Leftrightarrow (4\log x)^2 - 24(\log x)(\log y) + (3\log y)^2 = 0$

$\Leftrightarrow (4\log x - 3\log y)^2 = 0$

$\Leftrightarrow 4\log x - 3\log y = 0$

$\Leftrightarrow \frac{\log y}{\log x} = \frac{4}{3}$

$\Leftrightarrow \log_x y = \frac{4}{3}$

$\Leftrightarrow y = \underline{x^{\frac{4}{3}}}$.

3. $a = \log_5 16 \Leftrightarrow \log_{16} 5 = \frac{1}{a}$.

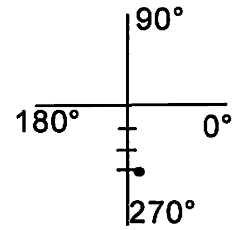
If $N = \log_{16} 5$, then $16^N = 5 \Leftrightarrow 2^{4N} = 5 \Leftrightarrow (2^3)^{\frac{4}{3}N} = 5 \Leftrightarrow 8^{\frac{4}{3}N} = 5 \Leftrightarrow \log_8 5 = \frac{4}{3}N$.

Thus, $\log_8 25 = \log_8(5^2) = 2\log_8 5 = 2\left(\frac{4}{3}N\right) = \frac{8}{3}\log_{16} 5 = \frac{8}{3}\cdot\frac{1}{a} = \underline{\frac{8}{3a}}$.

Detailed Solutions for GBML Meet 3 - DECEMBER 2017

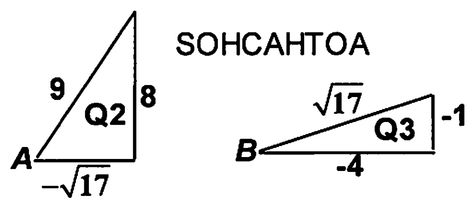
ROUND 5

1. $S = i + 2i^3 + 4i^5 + 6i^7 = i - 2i + 4i - 6i = -3i = \underline{\mathbf{3cis(270°)}}$.



2. Given: $\sin A = \frac{8}{9}$, $\cot A < 0$, $\tan B = \frac{1}{4}$, $\sin B < 0$

   These conditions uniquely determine the quadrants in which angles A and B lie.
   Now, applying the definitions of each trig function, we can substitute the numerical values.



$$(\tan A)(\sin B) - (\sec A)(\cos B) + \cos^2 B = \left(\frac{8}{-\sqrt{17}}\right)\left(\frac{-1}{\sqrt{17}}\right) - \left(\frac{9}{-\sqrt{17}}\right)\left(\frac{-4}{\sqrt{17}}\right) + \left(\frac{-4}{\sqrt{17}}\right)^2$$

$$= \frac{8}{17} - \frac{36}{17} + \frac{16}{17} = \underline{-\frac{\mathbf{12}}{\mathbf{17}}}.$$

3. Expanding the sums and differences,

   $\sin(x+30°) + \sin(x-30°) = \cos(x+60°) - \cos(x-60°) - \tan 240°$

   $\Leftrightarrow 2\sin x \cos 30° = -2\sin x \sin 60° - \sqrt{3}$

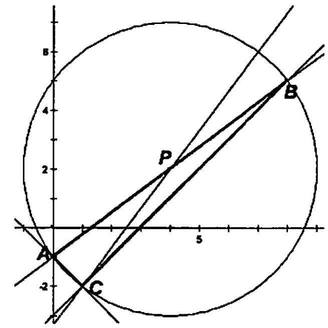
   $\Leftrightarrow \sqrt{3}\sin x = -\sqrt{3}\sin x - \sqrt{3}$

   $\Leftrightarrow \sin x = -\frac{1}{2}$

   $\Rightarrow x = \underline{\mathbf{210}}°, \underline{\mathbf{330}}°$.

Detailed Solutions for GBML Meet 3 - DECEMBER 2017

TEAM ROUND



1. $\begin{cases} L_1 : 3x-4y=4 \\ L_2 : 4x-3y=10 \end{cases} \Rightarrow (x,y)=P(4,2)$,

   $\begin{cases} L_1 : 3x-4y=4 \\ L_3 : x+y=-1 \\ L_4 : x-y=3 \end{cases} \Rightarrow A(0,-1),\ B(8,5),\ C(1,-2)$ and

   $L_2 : \{(x,y) \mid 4x-3y=10\}$.

   Applying the distance formula, $PA=PB=PC=5 \Rightarrow\ PA+PB+PC=\underline{\mathbf{15}}$.

   Note: L_3 and L_4 are perpendicular. P is the midpoint of the hypotenuse $\overline{AB}$ in right ΔABC. This makes P the center of the circle circumscribed about ΔABC which explains why each of the distances were the same. Each is a radius of the circumcircle!

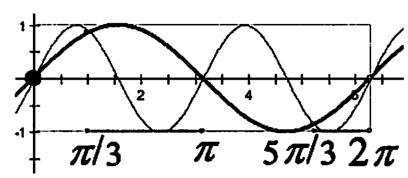
2. Consider the equation $\sin 2\theta = \sin\theta \Leftrightarrow 2\sin\theta\cos\theta - \sin\theta = 0 \Leftrightarrow \sin\theta(2\cos\theta - 1) = 0$.

   Thus, the functions defined by $y=\sin 2\theta$ and $y=\sin\theta$, over the interval $0 \le \theta < 2\pi$, intersect when $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$, i.e., when $x = 0, \pi, \frac{\pi}{3}$, and $\frac{5\pi}{3}$.

   Comparing the graphs, we require intervals where $y=\sin 2\theta$ (red/light) intersects or is below $y=\sin\theta$ (blue/dark).

   From the graph at the right, we see the required values are

   $\underline{\theta = \mathbf{0},\ \frac{\pi}{3} \le \theta \le \pi,\ \frac{5\pi}{3} \le \theta < 2\pi}$.



3. $AB = BC = 2\sqrt{3} \Rightarrow TO = \sqrt{3}$

   Since $\overline{TR}$ is an altitude of equilateral ΔABR, $TR = \frac{1}{2}(2\sqrt{3})\sqrt{3} = 3$.

   Thus, $OR = PO = 3-\sqrt{3} \Rightarrow PR = 6-2\sqrt{3}$.

   ΔPQO is a 30-60-90 right triangle.

   $$OQ = \frac{PO}{\sqrt{3}} = \left(\frac{3-\sqrt{3}}{\sqrt{3}}\right)\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}-3}{3} = \sqrt{3}-1$$

   $\Rightarrow QS = 2\sqrt{3}-2$.

   The required difference is $(6-2\sqrt{3})-(2\sqrt{3}-2) = \underline{\mathbf{8-4\sqrt{3}}}$.