GREATER BOSTON MATHEMATICS LEAGUE

MEET 3-DECEMBER 2018

ROUND 1-Algebra 1: Fractions and Word Problems

1. ______________________

2. ______________________

3. (__________ , __________)

1. Compute <u>all</u> ordered pairs of base 10 digits (a,b), where a and b have opposite parity, for which $\frac{10a+b}{10b+a}=\frac{4}{7}$.

2. For positive integers x and A, compute the minimum value of $A > 1000$ for which $\frac{1}{x}-\frac{1}{x+1}=\frac{1}{A}$.

3. A coat I would like to buy is going on sale at two different stores. On the first day of the sale in either store, the cost of the coat is \$100. In the first store, on subsequent days, the cost of the coat is always reduced by 20% of the cost from the previous day. In the second store, the cost of the coat is discounted x% on the second day and $(x + 16)$% on the third day. The cost of the coat in the first store on the 4^{th} day equals the cost of the coat on the 3^{rd} day in the second store. Let Y (in dollars and cents) denote the cost of the coat in the first store on the 4^{th} day. Compute the ordered pair (x, Y). Y must be expressed to two decimal places, rounded if necessary.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2018

ROUND 2-Coordinate Geometry of the Straight Line

1. (____________ , ____________)

2. ____________________________

3. ____________________________

1. Given: $A(k,2)$, $B(4,k)$

   If M, the midpoint of $\overline{AB}$, has coordinates $(2k+1, j)$, compute the ordered pair (k, j).

2. Given: $A(3,-1)$, $B(-1,5)$

   The triangle formed by the line $\overleftrightarrow{AB}$ and the coordinate axes has an area of k square units. Compute k.

3. Let O be the origin of the coordinate plane. Line $\overleftrightarrow{PQ}$ passes through (14, 140) with a slope of $\frac{7}{2}$. Compute the coordinates of <u>all</u> lattice points R on $\overleftrightarrow{PQ}$ in quadrant 2, such that the distance RO is an integer.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2018

ROUND 3- Geometry -Polygons: Area and Perimeter

1. The area is increaseddecreased by _____ square units.

2. ______________________________

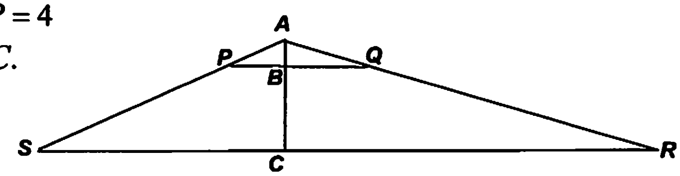
3. ______________________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. Square *ABCD* has sides of length 75. The lengths of a pair of parallel sides of *ABCD*are increased by 20%. The lengths of the other pair of parallel sidesof *ABCD*are decreased by 20%. The perimeter is unchanged, but the area has been changed by *k* square units. Circle the correct underlined word in the answer space and fill in the value of *k*.

2. Point *P* is located on the altitude drawn from vertex *A*to side $\overline{BC}$ in equilateral ΔABC. If the area of ΔABC is $36\sqrt{3}$ square units and the distance from *P* to $\overline{BC}$ is 8 units, compute the perimeter of ΔAPB.

3. Given: $\overline{PQ} \parallel \overline{SR}$, $\overline{AC} \perp \overline{SR}$, $AB = 2, BQ = 6$, $BP = 4$
If the area of trapezoid *PQRS* is 240, compute *BC*.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2018

ROUND 4-Algebra 2: Logs, Exponents Radicals and equations involving them

1. ______________________

2. ______________________

3. (____________ , ____________)

1. Compute $\log_4\left(8^{2018}\right)$.

2. Given: $y = 2^{x(x-1)}$

   For $x < \frac{1}{2}$, solve for x in terms of y.

3. Let $f(x) = 2^x + 100\left(2^{-x}\right)$ and $g(x) = 4^x$.

   These functions intersect at the point $P(a,b)$.

   Compute the ordered pair (a,b).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2018

ROUND 5- Trig Analysis and Complex Numbers in Trig Form

1. ______________________

2. (__________ , __________)

3. ______________________

1. Solve for x over $0° \le x \le 210°$: $\cos(3x - 4°) = \sin(x + 2°)$

2. The function $y = 2 - \cos\left(\frac{\pi}{4} - 3x\right)$ attains a maximum value at the point $P(h, k)$. Compute the ordered pair (h, k), where h has a *minimum* positive value.

3. The three cube roots of $-4\sqrt{2} + 4\sqrt{2}i$ are $\begin{cases} A cis\theta_1 \\ A cis\theta_2 \\ A cis\theta_3 \end{cases}$, where $0 \le \theta_1 < \theta_2 < \theta_3 < 2\pi$. Compute the ordered pair (A, θ_2).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2018

TEAM ROUND

3 pts.	1.	(__________ , __________)
3 pts.	2.	(______ , ______ , ______)
4 pts.	3.	______________________

1. Given: $P(4, 8)$, $Q(7, -1)$
   Compute the coordinates of the point on the line $\overleftrightarrow{PQ}$ closest to the origin.

2. An equilateral triangle *PBC* is constructed such that vertex *P* is in the interior of square *ABCD*.Let $(BC, PD) = (2x, y)$. The perimeter of *ABCD* is $a\left(\sqrt{b}+\sqrt{c}\right)y$. Compute the ordered triple of constants (a, b, c), where $b > c$.

3. In a local town election with three candidates, the number of votes Alexander received was 90% of the number Benedict received. The number of votes Carter received was 20 more than 80% of the number that Benedict received. Had 10 of the people who voted for Benedict and 5 of the people who voted for Carter been persuaded to vote for Alexander, Alexander would have won the town election by a <u>single</u> vote. There was a total of *k* votes cast in this election. Compute <u>all</u> possible values of *k*.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 - DECEMBER 2018

Answer Sheet

Round 1

1. $(1, 2), (3, 6)$
2. 1056
3. (20, \$51.20)
   Two digits after the decimal point are required, but the '\$' is not.

Round 2

1. $\left(\frac{2}{3}, \frac{4}{3}\right)$
2. $\frac{49}{12}$
3. $(-20, 21), (-24, 7)$

Round 3

1. decreased by 225
2. $14+6\sqrt{3}$ or $2\left(7+3\sqrt{3}\right)$
3. 8

Round 4

1. 3027
2. $x=\frac{1-\sqrt{1+4\log_2 y}}{2}$ or $\frac{1-\sqrt{\log_2\left(2y^4\right)}}{2}$
3. $(\log_2 5, 25)$
   Alternately, $(\log_4 25, 25)$ or $(2\log_4 5, 25)$

Round 5

1. 23°, 113°, 203°
2. $\left(\frac{5\pi}{12}, 3\right)$
3. $\left(2, \frac{11\pi}{12}\right)$

Team Round

1. (6, 2) (3 pts)
2. (2, 6, 2) (3 pts)
3. 47, 668 (4 pts)

Detailed Solutions for GBML Meet 3 - DECEMBER 2018

ROUND 1

1. $\dfrac{10a+b}{10b+a}=\dfrac{4}{7}\Leftrightarrow 70a+7b=40b+4a\Leftrightarrow 66a=33b\Leftrightarrow b=2a$.
   Since a and b have opposite parity, a must be odd and b must be even.
   Thus, $(a,b)=\underline{\mathbf{(1,2)}},\ \underline{\mathbf{(3,6)}}$.

2. $\dfrac{1}{x}-\dfrac{1}{x+1}=\dfrac{1}{A}\Leftrightarrow\dfrac{x+1-x}{x(x+1)}=\dfrac{1}{x(x+1)}=\dfrac{1}{A}\Rightarrow A=x(x+1)>1000$

   Since $2^{10}=1024$, we have $\left(2^5\right)^2=32^2=1024$.
   $31\cdot 32=32(32-1)=1024-32<1000\Rightarrow A_{min}=32\cdot 33=1024+32=\underline{\mathbf{1056}}$.

3.

Day	Cost in Store #1
1	100
2	$100\left(\frac{4}{5}\right)=80$
3	$80\left(\frac{4}{5}\right)=64$
4	$64(.8)=\$51.20$

An x% discount means I pay the original cost times $(1-x\%)=1-\dfrac{x}{100}=\dfrac{100-x}{100}$. Thus, the cost in store #2 on day 3 is $100\left(\dfrac{100-x}{100}\right)\left(\dfrac{84-x}{100}\right)=51.2\Rightarrow(100-x)(84-x)=5120$
$\Leftrightarrow 8400-184x+x^2=5120\Leftrightarrow x^2-184x+3280=0$
$\Leftrightarrow(x-20)(x-164)=0\Rightarrow x=20,\not{164}$
[The latter value is extraneous, since the discount can't exceed 100%.]
Thus, $(x,Y)=\underline{\mathbf{(20,\ \$51.20)}}$. The dollar sign is not required.

Detailed Solutions for GBML Meet 3 - DECEMBER 2018

ROUND 2

1. Since $M\left(\frac{k+4}{2},\frac{2+k}{2}\right)=(2k+1,j)$, we have $\frac{k+4}{2}=2k+1 \Rightarrow k+4=4k+2 \Rightarrow k=\frac{2}{3}$ and

   $j=\frac{2+\frac{2}{3}}{2}=\frac{4}{3}$

   Thus, $(k,j)=\underline{\left(\mathbf{\frac{2}{3},\frac{4}{3}}\right)}$.

2. Given: $A(3,-1)$, $B(-1,5)$

   The equation of $\overleftrightarrow{AB}$ is $(y-5)=-\frac{3}{2}(x+1) \Leftrightarrow 3x+2y-7=0$.

   Thus, the x-intercept is at $\left(\frac{7}{3},0\right)$ and the y-intercept is at $\left(0,\frac{7}{2}\right)$.

   The required area is $\frac{1}{2}\cdot\frac{7}{3}\cdot\frac{7}{2}=\underline{\mathbf{\frac{49}{12}}}$.

3. Start at (14, 140). Considering a slope of $m=\frac{7}{2}$, a decrease in the x-coordinate of at least 16 (and, correspondingly, a decrease of at least 56 in the y-coordinate) is required for the lattice point R to be in quadrant 2. It is required that x^2+y^2 generate a perfect square, so we are looking for a Pythagorean Triple (or multiple thereof).

x	y		Verdict	x	y		Verdict
14	140			-14	42	14(1,3,?)	NO
-2	84	2(1,42,?)	NO	-16	35		NO
-4	77		NO	-18	28	2(9,14,?)	NO
-6	70	2(3,35,?)	NO	-20	21	(20,21,**29**)	YES
-8	63		NO	-22	14	2(11,7,?)	NO
-10	56	2(5,28,?)	NO	-24	7	(24,7,**25**)	YES
-12	49		NO	-26	0	Exiting Q2	

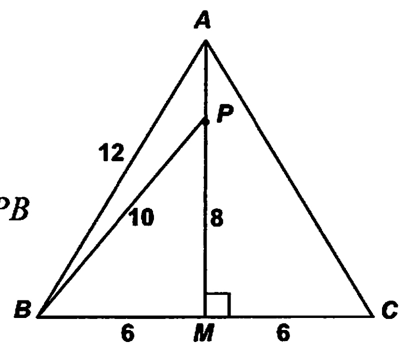
Thus, the possible coordinates of point R are $\underline{\mathbf{(-20,21)}}$ and $\underline{\mathbf{(-24,7)}}$.

Detailed Solutions for GBML Meet 3 - DECEMBER 2018

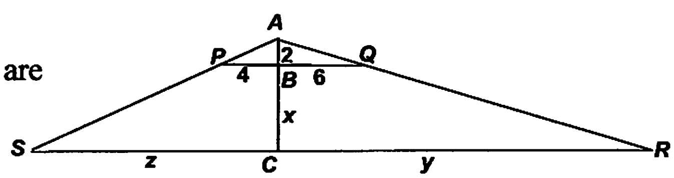
ROUND 3

1. Temporarily ignoring the given side length, the new rectangle will have dimensions $\left(x-\frac{1}{5}x\right)=\frac{4}{5}x$ and $\left(x+\frac{1}{5}x\right)=\frac{6}{5}x$, producing an area of $\frac{24}{25}x^2$ which is a **decrease** of $\frac{1}{25}$ of the original area. $k=\frac{1}{25}\cdot 75^2=3(75)=\underline{\mathbf{225}}$.
   Note: The original perimeter of the square was 4*x*. The rectangle's perimeter is $2\left(\frac{6x}{5}+\frac{4x}{5}\right)=\frac{20x}{5}=4x$, confirming there is no change in the perimeter.

2. Since the area of the equilateral triangle is $36\sqrt{3}$, we have
   $\frac{s^2\sqrt{3}}{4}=36\sqrt{3}\Rightarrow s^2=4(36)\Rightarrow s=12$.
   Since $PB=10$ and $AM=6\sqrt{3}$, we have the perimeter of ΔAPB is $10+12+\left(6\sqrt{3}-8\right)=\underline{\mathbf{14+6\sqrt{3}}}$ or $\underline{\mathbf{2\left(7+3\sqrt{3}\right)}}$.



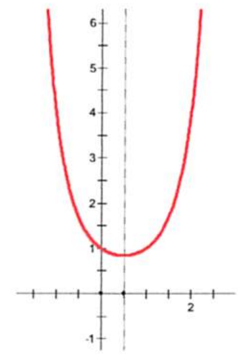
3. Let $(BC,CR,CS)=(x,y,z)$. By similar triangles, $\Delta ABQ\sim\Delta ACR$ and $\Delta ABP\sim\Delta ACS$, we have $\frac{BQ}{CR}=\frac{AB}{AC}$ and $\frac{BP}{CS}=\frac{AB}{AC}$, implying all three ratios are equal. Specifically,
   $\frac{2}{x+2}=\frac{6}{y}=\frac{4}{z}\Rightarrow y=3(x+2),\ z=2(x+2)$.
   $A_{trap}=\frac{1}{2}x(10+y+z)=\frac{1}{2}x(10+5(x+2))=240$
   $\Rightarrow x(5x+20)=480\Rightarrow x^2+4x-96=(x-8)(x+12)=0\Rightarrow x=BC=\underline{\mathbf{8}}$.



Detailed Solutions for GBML Meet 3 - DECEMBER 2018

ROUND 4

1. $\log_4\left(8^{2018}\right) = 2018\cdot\log_4 8 = 2018\cdot\log_4\left(2^3\right) = 3\cdot 2018\cdot\log_4 2 = 3\cdot 2018\cdot\frac{1}{2} = \underline{\mathbf{3027}}$.



2. $y = 2^{x(x-1)} \Leftrightarrow x(x-1) = \log_2 y \Leftrightarrow x^2 - x - \log_2 y = 0$

   Considering this a quadratic equation in *x*, we apply the quadratic formula.

   $x = \frac{1\pm\sqrt{1+4\log_2 y}}{2}$. Since $x<\frac{1}{2}$, $x = \underline{\frac{\mathbf{1-\sqrt{1+4\log_2 y}}}{\mathbf{2}}}$ or $\underline{\frac{\mathbf{1-\sqrt{\log_2\left(2y^4\right)}}}{\mathbf{2}}}$.

   The graph of $y = 2^{x(x-1)}$ is shown at the right.

   Notice the minimum value of *y* occurs when $x=\frac{1}{2}$, namely,

   $y=2^{-\frac{1}{4}}=\frac{1}{\sqrt[4]{2}}\cdot\frac{\sqrt[4]{8}}{\sqrt[4]{8}}=\frac{\sqrt[4]{8}}{2}\approx 0.841$.

3. $f(x)=g(x)\Leftrightarrow 2^x+100\left(2^{-x}\right)=4^x=\left(2^2\right)^x=2^{2x}=\left(2^x\right)^2$

   Thus, $\left(2^x\right)^2-2^x-\frac{100}{2^x}=0\Leftrightarrow\left(2^x\right)^3-\left(2^x\right)^2-100=0$.

   By inspection, if $y=2^x=5$, we have 125 – 25 – 100 = 0.

   By synthetic substitution, $\frac{y^3-y^2-100}{y-5}\Rightarrow$

	1	−1	0	−100
5	1	4	20	0

   The quadratic factor, $y^2+4y+20$, has only complex roots.

   Since $2^x=5\Rightarrow 4^x=25$, the point of intersection is $\underline{(a,b)=(\mathbf{\log_2 5,\ 25})}$.

   Alternately, $\log_2 5$ may be expressed as $2\log_4 5$, $\log_4 25$

Detailed Solutions for GBML Meet 3 - DECEMBER 2018

ROUND 5

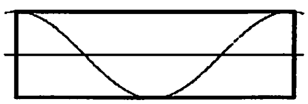
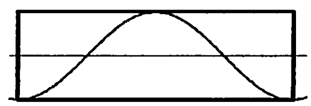
1. Since cosine is an even function, i.e., $\cos(-\theta)=\cos\theta$, and trig functions of coterminal angles are equal, $\cos(3x-4°)=\sin(x+2°) \Leftrightarrow \cos(3x-4°)=\pm\cos(90-(x+2°))$.

$$\Rightarrow 3x-4=\begin{Bmatrix}88-x\\-88+x\end{Bmatrix}+n(360°).$$

$$\Rightarrow \begin{cases}4x=92°+n(360°)\\2x=-84°+n(360°)\end{cases} \Rightarrow x=\begin{cases}23°+n(90°)\\-42°+n(180°)\end{cases} \Rightarrow \underline{\mathbf{23°,113°,203°,138°}}.$$

2. Since $-1\le\cos\theta\le1$, $2-(\pm1)$ generates a maximum value of +3, but for what value of x?

$y=-\cos\left(\frac{\pi}{4}-3x\right)+2$ isa "flipped" cosine function, i.e., reflected over the x-axis.One cycle

(period) of the cosine  has become .

The maximums usually occur at the beginning and end of a cycle, the minimum at the midpoint of a cycle, and the zeros at the quarter-section points. The zeros still occur $\frac{1}{4}$ and $\frac{3}{4}$ of the way through a cycle, but the locations of the maximums and minimums have swapped.Starting with the usual window, we can find its new location and size.As already noted, the centerline is at 2, the top of the window at 3 and the bottom at 1.

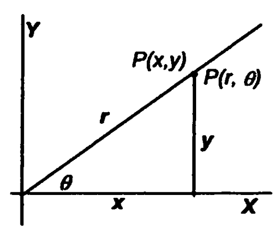
$0\le\frac{\pi}{4}-3x\le2\pi \Leftrightarrow -\frac{\pi}{4}\le-3x\le\frac{7\pi}{4} \Leftrightarrow \frac{\pi}{12}\ge x\ge-\frac{7\pi}{12}$ or $-\frac{7\pi}{12}\le x\le\frac{\pi}{12}$. Thus, the left side of the window (start of a cycle) is at $-\frac{7\pi}{12}$ and the right side of the window (end of a cycle) is at $\frac{\pi}{12}$.

The length of a cycle is $\frac{\pi}{12}-\left(-\frac{7\pi}{12}\right)=\frac{2\pi}{3}$. The maximum occurs in the middle of the window at

$\dfrac{-\frac{7\pi}{12}+\frac{\pi}{12}}{2}=-\frac{\pi}{4}$. Maxima occur once in every cycle. By adding $\frac{2\pi}{3}$, the minimum positive value ofx for which a maximum occurs isx $=-\frac{\pi}{4}+\frac{2\pi}{3}=\frac{5\pi}{12}$. Thus, $(x,y)=\underline{\left(\mathbf{\frac{5\pi}{12},3}\right)}$.

3. Converting to trig form, $-4\sqrt{2}+4\sqrt{2}i=4\sqrt{2}(-1+i)=rcis\theta$, where $x^2+y^2=r^2$ and $\tan\theta=\frac{y}{x}$.

Thus, $r^2=1+1=2\Rightarrow r=\sqrt{2}$ and $\tan\theta=\frac{1}{-1}=-1$.



θ is in quadrant 2 $\Rightarrow\theta=\frac{3\pi}{4}$. $-4\sqrt{2}+4\sqrt{2}i=4\sqrt{2}\cdot\sqrt{2}cis\left(\frac{3\pi}{4}\right)=8cis\left(\frac{3\pi}{4}\right)$

$(rcis\theta)^{\frac{1}{3}}=r^{\frac{1}{3}}cis\left(\frac{\theta}{3}\right)=8^{\frac{1}{3}}cis\frac{1}{3}\left(\frac{3\pi}{4}+2n\pi\right)\Rightarrow r=2,\ \theta=\frac{\pi}{4}+\frac{2n\pi}{3}$

$n=0\Rightarrow\theta_1=\frac{\pi}{4}$, $n=1\Rightarrow\theta_2=\frac{\pi}{4}+\frac{2\pi}{3}=\frac{11\pi}{12}$. The required ordered pair is $\underline{\left(\mathbf{2,\frac{11\pi}{12}}\right)}$.

Detailed Solutions for GBML Meet 3 - DECEMBER 2018

TEAM ROUND

1. Given: $P(4, 8)$, $Q(7, -1)$

   The slope of $\overleftrightarrow{PQ}$ is $m = \dfrac{8+1}{4-7} = -3 \Rightarrow m_{\perp} = +\dfrac{1}{3}$.

   The equation of $\overleftrightarrow{PQ}$ is $3x+y-k=0$.

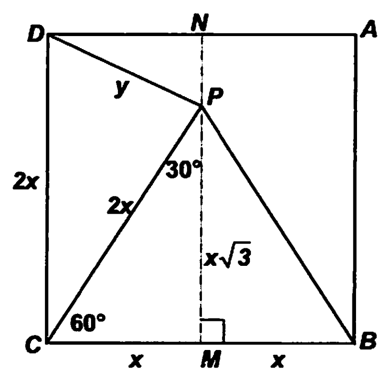
   Substituting the coordinates of *P*, we have $12 + 8 - k = 0 \Rightarrow k = 20$.

   The equation of the normal line (i.e., perpendicular) to $\overleftrightarrow{PQ}$, passing through the origin, is

   $(y-0) = \dfrac{1}{3}(x-0) \Leftrightarrow x = 3y$.

   Substituting, $3(3y)+y-20=0 \Rightarrow y=2 \Rightarrow (x,y) = \underline{\mathbf{(6,2)}}$.

2. In ΔPND, $(DN, PN) = \left(x, (2-\sqrt{3})x\right)$.



   $\Rightarrow y^2 = x^2 + \left(x\left(2-\sqrt{3}\right)\right)^2$.

   $\Rightarrow y^2 = x^2\left(1+4-4\sqrt{3}+3\right) = 4\left(2-\sqrt{3}\right)x^2$.

   $\Rightarrow y = 2x\sqrt{2-\sqrt{3}}$.

   We require that $2\sqrt{2-\sqrt{3}}$ be expressed in the form $\sqrt{b}-\sqrt{c}$.

   Squaring both sides, $4\left(2-\sqrt{3}\right) = b+c-2\sqrt{bc}$.

   Thus, we must have $b+c=8$ and $-2\sqrt{bc} = -4\sqrt{3} \Leftrightarrow \sqrt{bc} = 2\sqrt{3} \Leftrightarrow bc = 12$.

   Clearly $(b,c) = (6,2)$ or $(2,6)$. The negative values for *b* and *c*are rejected, since they would produce complex numbers for $\sqrt{b}$ and $\sqrt{c}$. Since $\sqrt{b}-\sqrt{c} > 0$, we have only $(b,c) = (6,2)$.

   Thus, $y = \left(\sqrt{6}-\sqrt{2}\right)x$, and the perimeter of *ABCD* is

   $8x = 8\dfrac{y}{\sqrt{6}-\sqrt{2}} \cdot \dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \dfrac{8\left(\sqrt{6}+\sqrt{2}\right)}{6-2}y = 2\left(\sqrt{6}+\sqrt{2}\right)y$ and $(a,b,c) = \underline{\mathbf{(2,6,2)}}$.

TEAM ROUND - continued

3. Let (A, B, C) denote the votes cast for Alexander, Benedict, and Carter, respectively. We were given $A = 0.9B$ and $C = 20 + 0.8B$. Let $x = A + B + C = 20 + 2.7B$ be the total votes cast for the three candidates. Then: $B = \frac{x-20}{2.7}$. Consider the following table of votes cast in the actual election and in the hypothetical election where A receives 10 of B's votes and 5 of C's votes.

X	*B*	*A*	*C*	*B*	*A*	*C*
20	0					
47	10	9	28	0	**24**	**23**
74	20	18	36	10	33	31
101	30	27	44	20	42	39
128	40	36	52	30	51	47
155	50	45	60	40	60	55

Notice in the second row ($x = 47$), A wins the election and beats C by 1 vote.
We are now looking for the case where A wins the election and beats B by one vote.
Notice in the last three rows, the gap between B and A is narrowing by 1 vote as we move down the table. Thus, in 19 more rows, A will beat B by a single vote.
In each row, there is a constant difference in the first 4 columns.
x, B, A and C are increasing by 27, 10, 9 and 8, respectively.
Thus, we can jump over the next 18 rows by adding 19 times these increments to the last entries in the first 4 columns.

x	*B*	*A*	*C*	*B*	*A*	*C*
155	50	45	60	40	60	55
+19(27) = 513	+19(10) = 190	+19(9) = 171	+19(8) = 152			
668	240	216	212	**230**	**231**	207

Thus, the total number of votes cast is either **<u>47</u>** or **<u>668</u>**.