GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2015

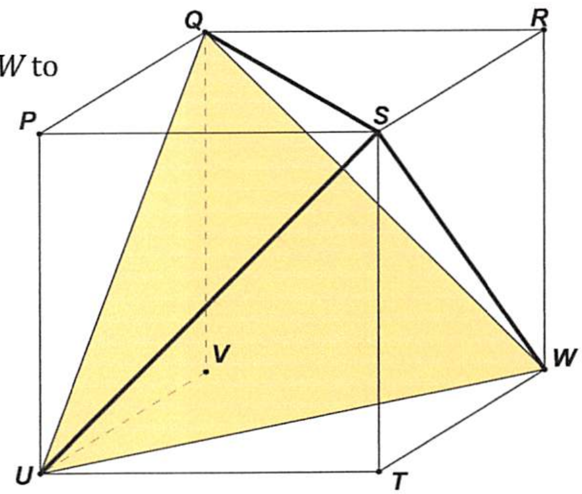
ROUND 1 – Volume and Surface Area of Solids

1. ______________________

2. ______________________

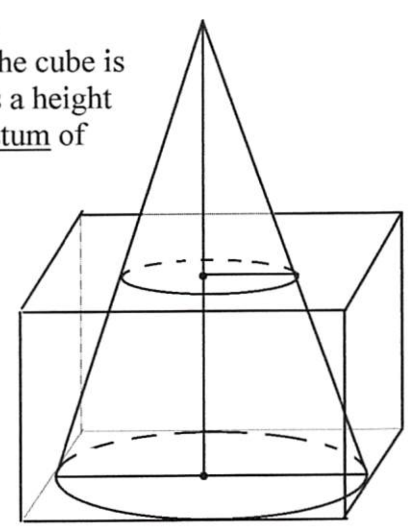
3. ______________________

1. The ratio of the surface area of tetrahedron $SQUW$ to the surface area of the cube $PQRSTUVW$ is $a : b$. Compute $\frac{a}{b}$.



2. The base of a regular hexagonal pyramid has a perimeter $6s$. The pyramid has a height equal to the sum of the lengths of two smaller diagonals of the base and the length of one longer diagonal of the base. Compute the volume of this pyramid in terms of s.

3. A right circular cone has its base inscribed in a face of a cube. The intersection of the interior of the cone and the interior of the cube is nonempty. The section of the cone not interior to the cube has a height equal to a side of the cube. The ratio of the volume of the frustum of the cone to the volume of the cube is $K : J$. Compute $\frac{K}{J}$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2015

ROUND 2 – Inequalities and Absolute Value

1. ______________________

2. ______________________

3. ______________________

1. Determine all values of x for which $\dfrac{\dfrac{1}{x}-\dfrac{3}{x^2}}{1-\dfrac{9}{x^2}} \geq 0$.

2. Compute <u>all</u> real values of x such that $|x+3|-|x-1|=x+1$.

3. Determine all values of x for which $|x^2-3x-1| \leq 3$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2015

ROUND 3 – Similar Polygons, Circles and Area

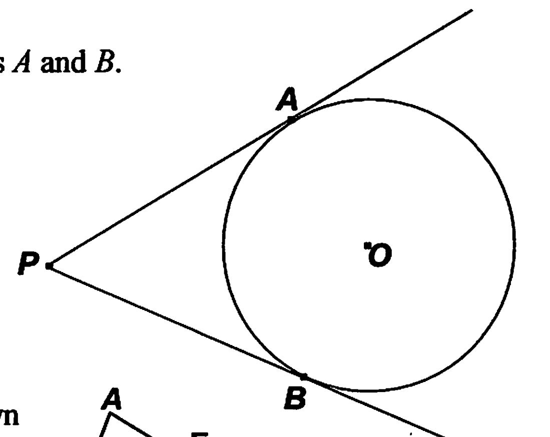
1. ______________________

2. ______________________

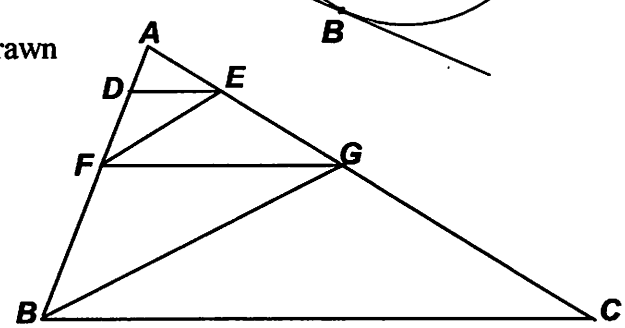
3. ______________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

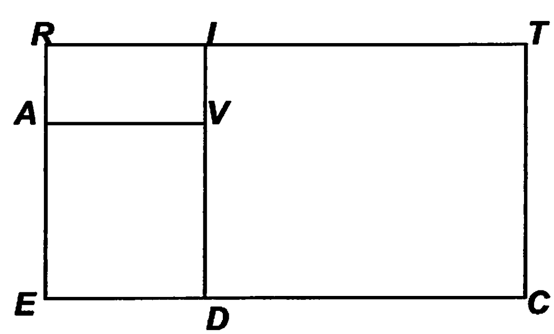
1. $\overline{PA}$ and $\overline{PB}$ lie on lines tangent to circle O at points A and B.
   Major arc $\overset{\frown}{AB}$ has a measure of 300°.
   The area of circle O is 48π.
   Compute the area of quadrilateral $PBOA$.



2. In ΔABC, two line segments $\overline{DE}$ and $\overline{FG}$ are drawn parallel to base $\overline{BC}$.
   D and F lie on $\overline{AB}$. E and G lie on $\overline{AC}$.
   Let $A(\ )$ denote area of.
   If $AD:DF:AB = 1:2:6$, compute the ratio $\left(A(\Delta DEF)+A(\Delta FGB)\right):A(\Delta ABC)$



3. Rectangle $RECT \sim RIDE \sim RAVI$
   If $RE = 8$, $RT > RE$, and the area of $RAVI$ is $\frac{8}{5^{12}}$, compute RT.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2015

ROUND 4 – Sequences and Complex Numbers

1. ______________________

2. ______________________

3. ______________________

1. Compute <u>all</u> possible values for x.

$$64+64x^2+64x^4+64x^6+......=100$$

2. The first term of an arithmetic progression is $1 + 2015i$.
The common difference is $2 - 13i$.
The sum of k terms is a real number. Compute this <u>unique</u> value of k.

3. The sequence 20, 30, 46, ab represents four two-digit numbers expressed in the same base, but it is <u>not</u> base 10. The first three form a <u>geometric</u> sequence, while the last three form an <u>arithmetic</u> sequence.
Compute the value of the two-digit number ab in base 10.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2015

ROUND 5 – Conics

1. (____________, ____________)

2. ________________________

3. ________________________

1. The equation of the hyperbola H is $x^2 - y^2 = 1$. The foci F_1 and F_2, and one endpoint of the minor axis form the vertices of a triangle. Compute the area of this triangle.

2. Given: $C_1 : x^2 + y^2 = 6y + 12 - 4x$
   The shortest distance from point $P(k, 0)$ to the circle C_1 is 4 units.
   Compute the unique <u>positive</u> value of k for which this is true.

3. The equation of the parabola P is $y^2 - 12x = 6y + 15$. The focus of P is the center of an ellipse, and the endpoints of the focal chord (latus rectum) is the major axis of the ellipse, while the vertex of P is one endpoint of the minor axis. Find the equation of the ellipse in $Ax^2 + Cy^2 + Dx + Ey + F = 0$ form, where A, C, D, E and F are integers and $A > 0$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2015

TEAM ROUND

3 pts.	1.	______________
3 pts.	2.	______________
4 pts.	3.	______________

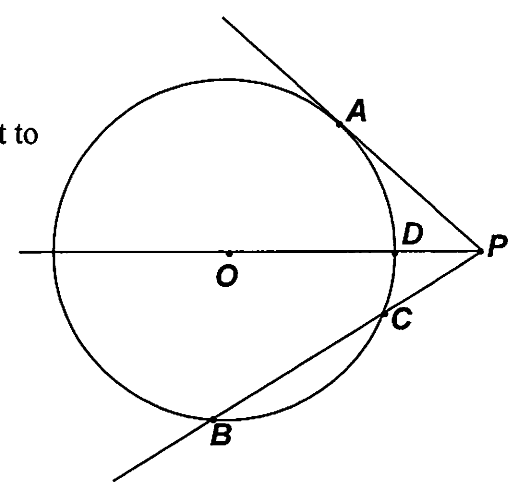
1. Compute <u>all</u> such x such that: $\left|\dfrac{x-1}{x+1}\right| + |x+2| = 3$.

2. Given the following sequence of positive terms:

$$\frac{2}{4A-1}, \frac{1}{2A+2}, \frac{2}{8A+3}, H, \frac{1}{X+1}, \frac{5}{3X-1}, P$$

The first 4 terms form a harmonic sequence and H is the first term of a geometric sequence of 4 terms. Compute all possible quadruples (A,H,X,P).

3. $\overline{PA}$ is tangent to circle O at point A and $\overline{PCB}$ is a secant to circle O (intersecting circle O at points B and C). $PD = 3$, $AB = 2\sqrt{13}$, $BC = 2PC$ and $m\measuredangle APB = 60°$. Compute the radius of circle O.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2015

Answer Sheet

Round 1

1. $\dfrac{\sqrt{3}}{3}$

2. $s^3\left(3+\sqrt{3}\right)$

3. $\dfrac{7\pi}{48}$

Round 2

1. $x>-3$ and $x\neq 0,+3$

2. $-5,-1,3$

3. $-1\leq x\leq 1,\ 2\leq x\leq 4$

Round 3

1. $16\sqrt{3}$

2. $\dfrac{11}{36}$

3. 10000

Round 4

1. $\pm\dfrac{3}{5}$

2. 311

3. 72

Round 5

1. $\sqrt{2}$

2. $6\sqrt{2}-2$

3. $4x^2+y^2-8x-6y-23=0$

Team Round

1. $0,1,-2,-3$ (3 pts)

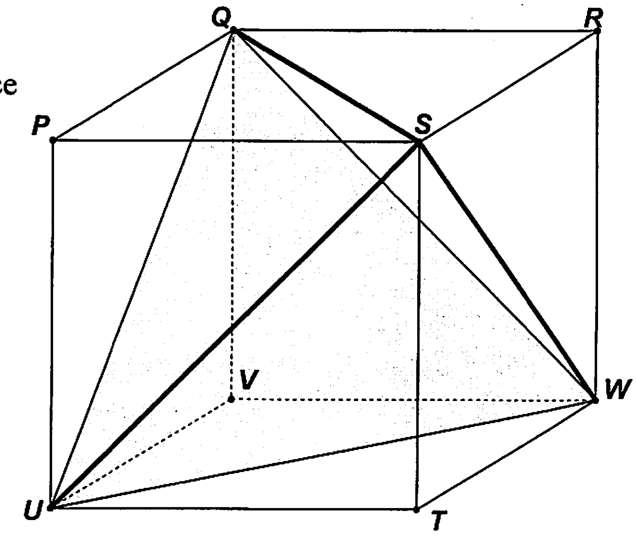
2. $\left(\dfrac{3}{2},\dfrac{1}{10},3,\dfrac{25}{16}\right),\left(\dfrac{3}{2},\dfrac{1}{10},1,\dfrac{25}{2}\right)$ (3 pts)

3. $\dfrac{1}{2}\left(13+4\sqrt{3}\right)$ (4 pts)

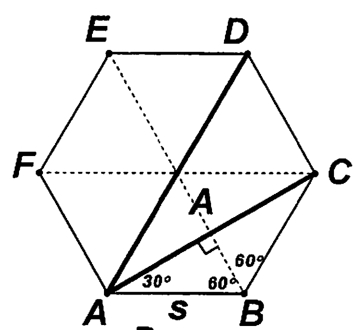
Detailed Solutions for GBML Meet 4 – JANUARY 2015

ROUND 1

1. Notice that the 6 edges of the tetrahedron are the face diagonals of the cube. If the edges of the cube are 1 unit, then the edges of the tetrahedron are $\sqrt{2}$ units.
   The surface area of the cube is 6 units^2.
   The surface area of the tetrahedron is
   $$4\left(\frac{\left(\sqrt{2}\right)^2\sqrt{3}}{4}\right)=2\sqrt{3}$$
   Therefore, the required ratio is $\frac{2\sqrt{3}}{6}=\underline{\frac{\sqrt{3}}{3}}$.

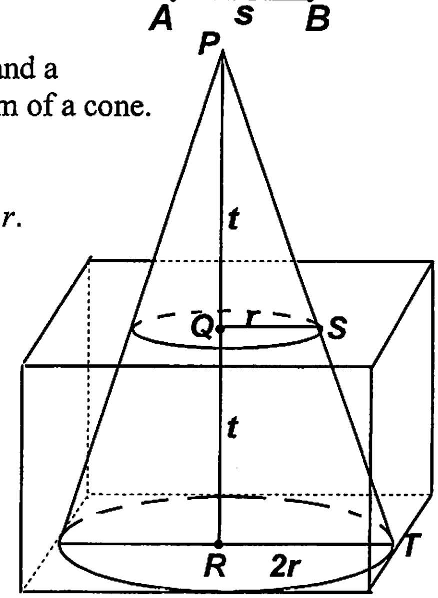


2. The short diagonal of the base has length $s\sqrt{3}$ and the longer diagonal has length $2s$. The volume of the pyramid is
   $$V=\frac{1}{3}Bh=\frac{1}{3}\cdot\frac{6s^2\sqrt{3}}{4}\left(2s\sqrt{3}+2s\right)=s^2\sqrt{3}\left(s\sqrt{3}+2s\right)=\underline{s^3\left(3+\sqrt{3}\right)}$$



3. The larger cone is comprised of two parts – a smaller cone (top) and a truncated cone (bottom). The latter is officially called the frustum of a cone. The frustum is the part of the larger cone interior to the cube.
   Let $4r$ denote the side of the cube. Then:
   $RT=2r$, $t=4r$ and, by similar triangles (ΔPQS and ΔPRT), $QS=r$.

   V(cube) = $(4r)^3=64r^3$
   V(frustum) = $\frac{1}{3}\pi(2r)^2(2t)-\frac{1}{3}\pi r^2 t=\frac{7}{3}\pi tr^2=\frac{28}{3}\pi r^3$
   Thus, $\frac{\frac{28}{3}\pi r^3}{64r^3}=\frac{28\pi}{64\cdot 3}=\underline{\frac{7\pi}{48}}$.



Detailed Solutions for GBML Meet 4 – JANUARY 2015

ROUND 2

1. $$\frac{\frac{1}{x}-\frac{3}{x^2}}{1-\frac{9}{x^2}}\ge 0 \Leftrightarrow \frac{\frac{1}{x}-\frac{3}{x^2}}{1-\frac{9}{x^2}}\ge 0 \Leftrightarrow \frac{\frac{x-3}{x^2}}{\frac{x^2-9}{x^2}} \Rightarrow x\ne 0, \pm 3$$

$$\frac{\frac{x-3}{x^2}}{\frac{x^2-9}{x^2}}=\frac{x-3}{x^2-9}=\frac{1}{x+3}\ge 0 \Rightarrow x>-3$$

Applying the restrictions above, we have $x>-3$ and $x\ne 0,+3$

Alternately expressed, $-3<x<0$ or $0<x<3$ or $x>3$.

Using interval notation, $(-3,0)\cup(0,3)\cup(3,\infty)$.

2. To eliminate the absolute value, the critical points to consider are -3 and $+1$. These numbers divide the number line into 3 regions and an equivalent equation can be written for each interval.

Case 1: $x<-3$

$(-x-3)-(1-x)=x+1 \Leftrightarrow x+1=-4 \Leftrightarrow x=\underline{-5}$

(Since this value is in the interval specified, it will check.)

Case 2: $-3<x<1$

$(x+3)-(1-x)=x+1 \Leftrightarrow 2x+2=x+1 \Leftrightarrow x=\underline{-1}$

Case 3: $x>1$

$(x+3)-(x-1)=x+1 \Leftrightarrow x=\underline{3}$

3. $|x^2-3x-1|\le 3 \Leftrightarrow -3\le x^2-3x-1\le +3$

$\Leftrightarrow -3\le x^2-3x-1$ and $x^2-3x-1\le +3$

$\Leftrightarrow x^2-3x+2\ge 0$ and $x^2-3x-4\le 0$

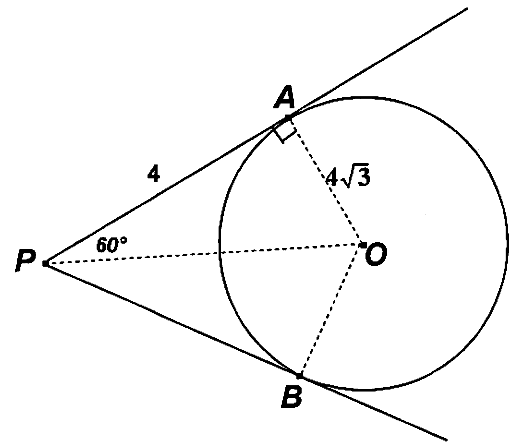
$\Leftrightarrow (x-2)(x-1)\ge 0$ and $(x-4)(x+1)\le 0$

$\Leftrightarrow (x\le 1 \text{ or } x\ge 2)$ and $-1\le x\le 4$

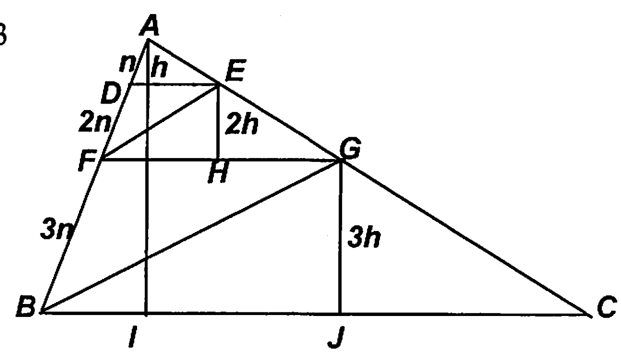
Taking the intersection, we have $\underline{\mathbf{-1\le x\le 1 \text{ or } 2\le x\le 4}}$.

Detailed Solutions for GBML Meet 4 – JANUARY 2015

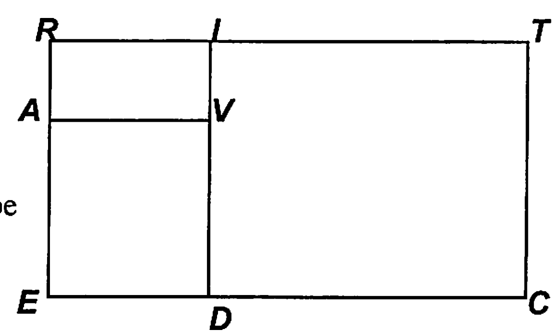
ROUND 3



1. $\pi r^2 = 48\pi \Rightarrow r = 4\sqrt{3}$
   Minor arc $\overset{\frown}{AB}$ measures 60°
   $m(\measuredangle P) = \frac{1}{2}(300° - 60°) = 120°$
   Area($PBOA$) = 2· Area(ΔAPO) = $2\left(\frac{1}{2}\cdot 4\cdot 4\sqrt{3}\right) = \underline{\mathbf{16\sqrt{3}}}$.



2. $AD:DF:AB = 1:2:6 \Leftrightarrow AD:DF:FB = 1:2:3$
   Since $\Delta ADE \sim \Delta AFG \sim \Delta ABC$,
   $EH:GJ:AI = 2:3:6$ and
   $DE:FG:BC = 1:3:6$
   Therefore, $\left(A(\Delta DEF) + A(\Delta FGB)\right) : A(\Delta ABC)$
   $= \left(\frac{1}{2}\cdot b\cdot 2h + \frac{1}{2}\cdot 3b\cdot 3h\right) : \left(\frac{1}{2}\cdot 6b\cdot 6h\right)$
   $= \frac{11bh}{36bh} = \underline{\mathbf{\frac{11}{36}}}$



3. The lengths of the sides of the similar rectangles must be in a $8:x$ ratio.
   Consequently, their areas must be in an $8^2:x^2$ ratio and successive areas can be found by multiplying the original area by $\frac{64}{x^2}$. Therefore, the area of *RAVI* will be
   $8x\cdot\left(\frac{64}{x^2}\right)^2 = \frac{8^5}{x^3} = \frac{8}{5^{12}} \Rightarrow$
   $x^3 = 8^4\cdot 5^{12} = 2^{12}\cdot 5^{12} = 10^{12} \Rightarrow x = \underline{\mathbf{10000}}$.

Detailed Solutions for GBML Meet 4 – JANUARY 2015

ROUND 4

1. $64+64x^2+64x^4+64x^6+\ldots\ldots=100 \Leftrightarrow 1+x^2+x^4+x^6+\ldots\ldots=\frac{100}{64}=\frac{25}{16}$

   Clearly, the left side of the equation is an infinite geometric series with a common multiplier of x^2, which will converge as long as $|x^2|=x^2<1$.

   $$1+x^2+x^4+x^6+\ldots\ldots=\frac{a}{1-r}=\frac{1}{1-x^2}=\frac{25}{16} \Rightarrow 1-x^2=\frac{16}{25} \Leftrightarrow x^2=\frac{9}{25} \Rightarrow x=\underline{\pm\frac{3}{5}}$$

2. The sum of n terms in an arithmetic progression with first term a and a common difference of d is given by $S=\frac{n}{2}(2a+(n-1)d)$.

   For $a = 1 + 2015i$ and $d = 2 - 13i$, the sum of k terms will be

   $S_k=\frac{k}{2}\left(\underline{2+4030i+(k-1)(2-13i)}\right)$

   If this product is to be real, the coefficient of i in the underlined expression must be 0.

   Expanding and collecting terms with a factor of i, we have $(4043-13k)i$

   Thus, $4043 - 13k = 0 \Rightarrow k = \underline{\mathbf{311}}$.

   It's interesting to note that the $S_{311}=311^2$.

3. If the four values are in base n, then we have a geometric sequence $2n$, $3n$, $4n+6$ and the common multiplier is $r=\frac{3n}{2n}=\frac{(4n+6)}{3n} \Rightarrow 9n^2=8n^2+12n \Rightarrow n(n-12)=0 \Rightarrow n=12$

   In base 10, the sequence is 24, 36, 54, $12a+b$. Therefore, $12a+b-54=18 \Rightarrow 12a+b=\underline{\mathbf{72}}$.

   FYI: Remember that $12a+b$ is the base 10 representation of the base 12 integer *ab*.

   Since a and b are digits in base 12, the only possible values are $a = 6$ and $b = 0$.

   Check: In the sequence 24, 36, 54, 72, the first three terms form a G.S. with a common multiplier of $\frac{3}{2}$ and the last three terms form an A. S. with a common difference of 18.

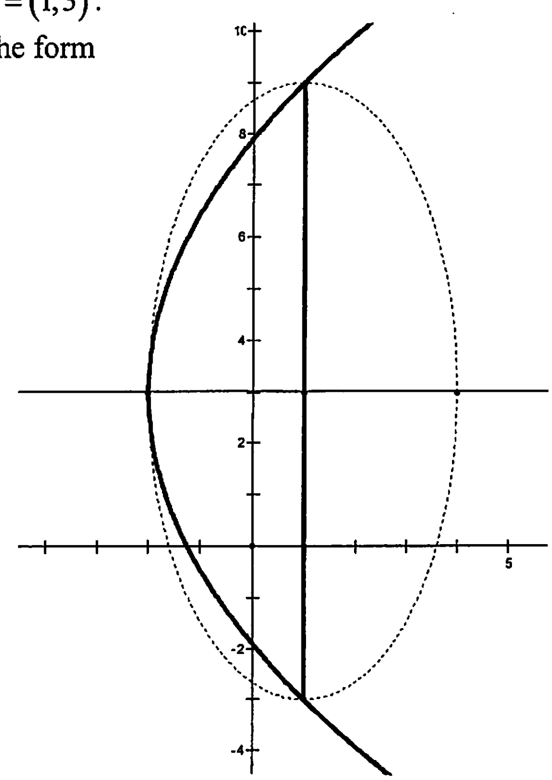
Detailed Solutions for GBML Meet 4 – JANUARY 2015

ROUND 5

1. $x^2 - y^2 = 1 \Rightarrow a = b = 1 \Rightarrow c = \sqrt{2}$. The hyperbola is origin-centered, opens left-right, and has foci at $\left(\pm\sqrt{2},0\right)$. The endpoints of the minor axis are $\left(0,\pm 1\right)$. Therefore, regardless of which endpoint is the third vertex of the triangle, the area of the required triangle is $\frac{1}{2}\cdot 2\sqrt{2}\cdot 1 = \underline{\sqrt{2}}$.

2. $x^2 + y^2 = 6y + 12 - 4x \Leftrightarrow (x+2)^2 + (y-3)^2 = 12 + 4 + 9 = 25$
   Thus, C_1 is a circle with center at (–2, 3) and radius of 5.
   The shortest distance lies along the line segment connecting (k, 0) and the center (–2, 3).
   Thus, we have $\sqrt{(k+2)^2 + 3^2} - 5 = 4$.
   $\Rightarrow (k+2)^2 = (4+5)^2 - 9 = 72 \Rightarrow k = \underline{\mathbf{-2+6\sqrt{2}}}$ $(-2-6\sqrt{2} < 0$ is extraneous).

3. $y^2 - 12x = 6y + 15 \Leftrightarrow \left(y^2 - 6y\underline{+9}\right) = 12x + 15\underline{+9} \Leftrightarrow (y-3)^2 = 12(x+2)$
   Therefore, the parabola P has a vertex at $(-2,3)$ and the distance to the focus, usually denoted p, satisfies $4|p| = 12$.
   Since the parabola opens to the right, $p = +3$. Since the axis of symmetry is $y = 3$, the coordinates of the endpoints of the focal chord are $(-2+3, 3+2\cdot 3) = (1,9)$ and $(-2+3, 3-2\cdot 3) = (1,-3) \Rightarrow a = 6, b = 3$ and $(h,k) = (1,3)$.
   The ellipse is vertical and its equation must be of the form
   $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$. Substituting,
   $\frac{(x-1)^2}{3^2} + \frac{(y-3)^2}{6^2} = 1 \Rightarrow 4(x-1)^2 + (y-3)^2 = 36$
   $\Rightarrow \underline{\mathbf{4x^2 + y^2 - 8x - 6y - 23 = 0}}$.



Detailed Solutions for GBML Meet 4 – JANUARY 2015

TEAM ROUND

1. Critical points at $-2, \pm 1$ require 4 cases to generate equivalent equations without absolute value.

   Case 1: $x < -2$

   $$\frac{-(x-1)}{-(x+1)} + (-x-2) = 3 \Rightarrow 1 - x + (x+2)(x+1) = -3x - 3$$

   $$\Leftrightarrow x^2 + 5x + 6 = (x+2)(x+3) = 0 \Rightarrow x = \cancel{-2}, \underline{-3}$$

   Case 2: $-2 \le x < -1$

   $$\frac{-(x-1)}{-(x+1)} + (x+2) = 3 \Rightarrow x - 1 + (x+2)(x+1) = 3x + 3$$

   $$\Leftrightarrow x^2 + x - 2 = (x+2)(x-1) = 0 \Rightarrow x = \underline{-2}, \cancel{1}$$

   Case 3: $-1 < x \le 1$

   $$\frac{-(x-1)}{(x+1)} + (x+2) = 3 \Rightarrow 1 - x + (x+2)(x+1) = 3x + 3$$

   $$\Leftrightarrow x^2 - x = x(x-1) = 0 \Rightarrow x = \underline{\mathbf{0, 1}}$$

   Case 4: $x > 1$

   $$\frac{(x-1)}{(x+1)} + (x+2) = 3 \Rightarrow x - 1 + (x+2)(x+1) = 3x + 3$$

   $$\Leftrightarrow x^2 + x - 2 = (x+2)(x-1) = 0 \Rightarrow x = \cancel{-2, 1}$$

   Therefore, $x = \underline{\mathbf{-3, -2, 0, 1}}$. Note that the important thing was that the intervals included all real numbers (except -1), but did not overlap.

2. If $\frac{2}{4A-1}, \frac{1}{2A+2}, \frac{2}{8A+3}, H$ is a harmonic sequence, then $\frac{4A-1}{2}, 2A+2, \frac{8A+3}{2}, \frac{1}{H}$ is an arithmetic sequence with a common difference of

   $$d = (2A+2) - \left(\frac{4A-1}{2}\right) = \left(\frac{8A+3}{2}\right) - (2A+2) \Rightarrow 4A + 4 - 4A + 1 = 8A + 3 - 4A - 4$$

   $\Rightarrow 5 = 4A - 1 \Rightarrow A = \frac{3}{2}$, resulting in the sequence $\frac{5}{2}, 5, \frac{15}{2}, \frac{1}{H}$. $d = \frac{5}{2} \Rightarrow \frac{1}{H} = 10 \Rightarrow H = \frac{1}{10}$

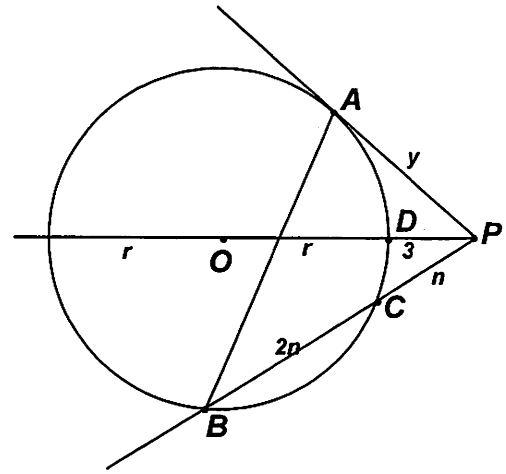
   If $\frac{1}{10}, \frac{1}{X+1}, \frac{5}{3X-1}, P$ is a geometric sequence, then

   $$r = \frac{10}{X+1} = \frac{5(X+1)}{3X-1} \Rightarrow 5(X+1)^2 = 30X - 10 \Rightarrow X^2 - 4X + 3 = (X-1)(X-3) = 0 \Rightarrow X = 1, 3$$

   $$X = 1 \Rightarrow r = 5 \Rightarrow P = \frac{25}{2}, \quad X = 3 \Rightarrow r = \frac{5}{2} \Rightarrow \frac{1}{10}, \frac{1}{4}, \frac{5}{8}, \ P = \frac{25}{16}$$

   Thus, $(A, H, X, P) = \underline{\left(\mathbf{\frac{3}{2}, \frac{1}{10}, 1, \frac{25}{2}}\right)}, \ \underline{\left(\mathbf{\frac{3}{2}, \frac{1}{10}, 3, \frac{25}{16}}\right)}$.

Detailed Solutions for GBML Meet 4 – JANUARY 2015



TEAM ROUND

3. $y^2 = 3(3+2r) = n(3n) \Rightarrow \begin{cases} n^2 = 2r+3 \\ y^2 = 3n^2 \end{cases} \Rightarrow \begin{cases} n = \sqrt{2r+3} \\ y = \sqrt{3}\cdot\sqrt{2r+3} \end{cases}$

Using the Law of Cosines in ΔPAB,

$$\left(2\sqrt{13}\right)^2 = y^2 + (3n)^2 - 2(y)(3n)\cos(60°)$$

$$52 = 3(2r+3) + 9(2r+3) - 2\sqrt{3}\cdot\sqrt{2r+3}\cdot 3\sqrt{2r+3}\cdot\frac{1}{2}$$

$$52 = 12(2r+3) - 3\sqrt{3}\cdot(2r+3)$$

$$52 = 3\left(4-\sqrt{3}\right)(2r+3)$$

$$r = \frac{1}{2}\left(\frac{52}{3\left(4-\sqrt{3}\right)}\cdot\frac{4+\sqrt{3}}{4+\sqrt{3}} - 3\right) = \frac{1}{2}\left(\frac{52\left(4+\sqrt{3}\right)}{13} - 3\right) = \underline{\mathbf{\frac{1}{2}\left(13+4\sqrt{3}\right)}}$$