GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 - JANUARY 2017

ROUND 1 - Volume and Surface Area of Solids

1. ______________________

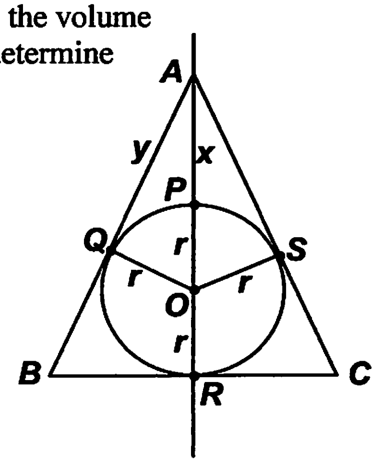
2. ______________________

3. ______________________

1. Cube C has edges of length 2. Cube C' is formed by connecting the centers of each of the faces of C. Compute the surface area of C'.

2. A plane passes though a pyramid parallel to its square base forming a frustum, a truncated pyramid. Compute the volume of this frustum, if the upper and lower bases have areas of 16 square units and 100 square units, respectively, and the distance between them is 6 units.

3. Circle O with radius r is inscribed in isosceles triangle ABC whose base $\overline{BC}$ has length 2. Let Q, R and S be points of tangency and $(AP, AQ) = (x, y)$.
Rotating ΔABC and circle O about the vertical axis produces a sphere inscribed in a cone. If the volume of the sphere is equal to the volume of the region inside the cone and outside the sphere, then determine the ordered pair (x, y), strictly in terms of r.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 - JANUARY 2017

ROUND 2 - Inequalities and Absolute Value

1. ______________________

2. ______________________

3. ______________________

1. For what real value(s) of x is the following statement true?

$$|x|^3 + 3|x|^2 + 3|x| - 63 = 0$$

2. Determine the minimum integer value of the constant A for which there are at least 2017 integer x-values which satisfy $|2x - 5| < A$.

3. Determine all real values of x which make the following statement true:

$$8x + \frac{27}{x^2} > 4 - \frac{6}{x} + \frac{9}{x^2}$$

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 3 - Similar Polygons, Circles and Area

1. ______________________________

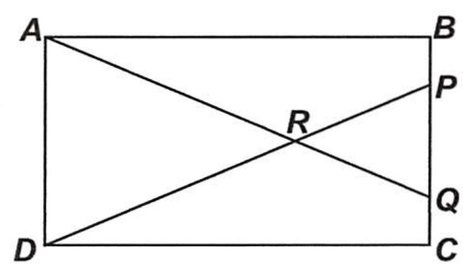
2. ____________ : ____________

3. ______________________________

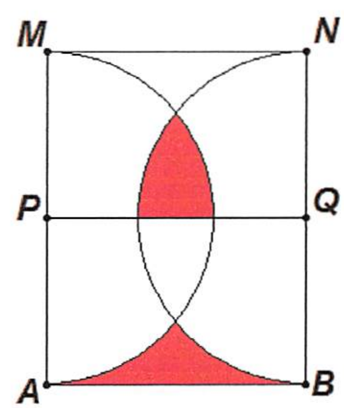
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. Semi-circles drawn on each side of a triangle have areas 9π, 16π, and 25π. Compute the area of the triangle.

2. In rectangle $ABCD$, $\frac{BP}{PC} = \frac{3}{7}$ and $\frac{QC}{QB} = \frac{4}{11}$. Compute the ratio of the area of ΔPQR to the area of ΔDAR.



3. Radii $\overline{PM}$ and $\overline{QN}$ are congruent and intersect parallel tangents $\overline{MN}$ and $\overline{AB}$. The shaded regions have equal areas and $PQ = 1$. Compute the area of quadrilateral $PMNQ$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 - JANUARY 2017

ROUND 4 - Sequences and Complex Numbers

1. ______________________

2. ______________________

3. ______________________

1. Given: $\left(1-\frac{a}{1}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{6}\right)=-a$.

   Compute $1+a+a^2+a^3+a^4+\ldots$ or indicate that the sum does not exist (DNE).

2. $\left(\frac{a}{b}\right)_{(\text{base }7)}$ is a simplified fraction equivalent to $1.6\bar{1}_{(\text{base }9)}$

   Compute the ordered pair (a,b).

3. Compute <u>all</u> ordered pairs of <u>complex</u> numbers (j,k) which satisfy the following system:

$$\begin{cases} j^3+k^3=100 \\ j^2k+jk^2=300 \end{cases} \text{ and } j+k \text{ is a real number.}$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 - JANUARY 2017

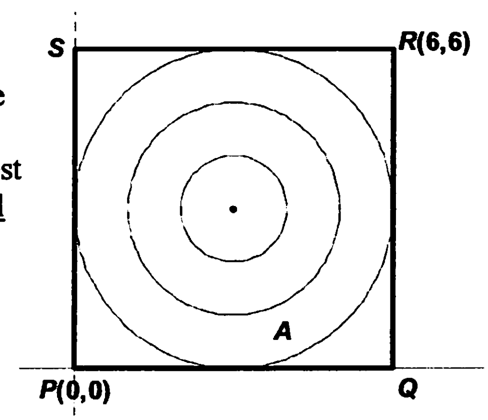
ROUND 5 - Conics

1. ______________________________

2. ______________________________

3. ______________________________

1. Three concentric circles centered at (3, 3) with radii of lengths 1, 2, and 3 are drawn inside a square whose side has length 6 as shown in the diagram at the right.
A line passing through (2,1) and a point in ring A farthest from the origin has an x-intercept at $(h, 0)$. Compute <u>all</u> possible values of h.



2. The horizontal line $y = 2$ intersects the parabola P whose equation is $(y-1) = \frac{x^2}{8}$ in points Q and R. A horizontal line $y = k$ intersects P in points S and T so that $ST = 2QR$. Compute k.

3. The distance from $P(6,4)$ to the fixed point $F(10,4)$ is half its distance from the vertical line $x = n$, where $n < 0$. The locus of all such points is a horizontal ellipse. P is the leftmost point on this ellipse. Point $Q(q_1, 4)$ is the rightmost point on the ellipse and point $R(r_1, r_2)$ is the uppermost point. Compute the ordered quadruple (n, q_1, r_1, r_2).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 - JANUARY 2017

TEAM ROUND

3 pts. 1. ____________________

3 pts. 2. ____________________

4 pts. 3. ____________________

1. The lengths of the sides of a triangle are integers and its area is also an integer. If one side has length 21 and the perimeter is 48, compute all possible lengths of the <u>shortest</u> side.

2. Compute the <u>minimum</u> <u>integer</u> value of n for which $2n+1-\sqrt{4n^2+4n}<0.01$.

3. The vertices of a triangle are $P(-1,-t)$, $Q(5,-5)$, and $W(3t,-3)$. Compute <u>all</u> possible values of t so that the area of ΔPQW is 50.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 - JANUARY 2017

Answer Sheet

Round 1

1. 12
2. 312
3. $\left(8r^3-2r,8r^4\right)$

Round 2

1. ± 3
2. 2018
3. $-1<x<0,\ x>0$ (or equivalent)

Round 3

1. 48
2. $\frac{169}{900}$
3. $\frac{2}{\pi}$

Round 4

1. $\frac{5}{6}$
2. (232, 132)
3. $\left(5+i\sqrt{5},5-i\sqrt{5}\right),\left(5-i\sqrt{5},5+i\sqrt{5}\right)$

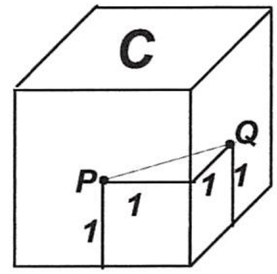
Round 5

1. $\frac{5}{4}$
2. 5
3. $\left(-2,22,14,14+4\sqrt{3}\right)$

Team Round

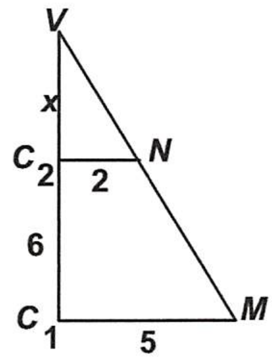
1. 10 (3 pts)
2. 25 (3 pts)
3. $-3,\frac{29}{3}$ (4 pts)

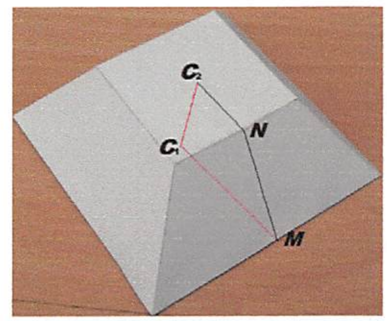
Detailed Solutions for GBML Meet 4 - JANUARY 2017



1. Clearly, $PQ = \sqrt{2}$.
   Thus, the surface area of C', the cube connecting the centers of the faces of C is $6\left(\sqrt{2}\right)^2 = \underline{\mathbf{12}}$.

2. Let V be the vertex of the pyramid.
   Let C_1 and C_2 be the centers of the base and the cross section respectively. Let M and N be midpoints of edges of the base and the cross section located on the same face of the pyramid.





$$\Delta VC_2N \sim \Delta VC_1M \Rightarrow \frac{x}{x+6} = \frac{2}{5} \Rightarrow 5x = 2x + 12 \Rightarrow x = 4$$

Thus, the volume of the frustum is

$$\frac{1}{3}(100)(10) - \frac{1}{3}(16)(4) = \frac{1}{3}(1000-64) = \frac{1}{3}(936) = \underline{\mathbf{312}}.$$

3. Sphere $= \dfrac{4}{3}\pi r^3$ Cone $= \dfrac{1}{3}\pi(x+2r)\cdot 1^2$

   Therefore,

   $$\frac{4}{3}\pi r^3 = \frac{1}{3}\pi(x+2r) - \frac{4}{3}\pi r^3 \Rightarrow 8r^3 = x + 2r \Rightarrow x = 8r^3 - 2r$$

   In ΔBAR, $(y+1)^2 = 1 + (x+2r)^2$

   In ΔAQO, $y^2 = (x+r)^2 - r^2 = x^2 + 2xr$

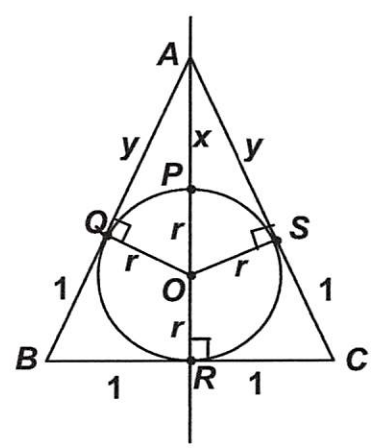
   Subtracting, $2y + 1 = 1 + x^2 + 4xr + 4r^2 - x^2 - 2xr$

   $\Leftrightarrow 2y = 2xr + 4r^2$

   $\Leftrightarrow y = xr + 2r^2$

   Substituting for x, $y = (8r^3 - 2r)r + 2r^2 = 8r^4$

   Thus, $(x, y) = \underline{\left(\mathbf{8r^3 - 2r, 8r^4}\right)}$.



Detailed Solutions for GBML Meet 4 - JANUARY 2017

ROUND 2

1. The only rational solutions will be factors of 63. Clearly, ± 1 fail. By inspection, any *even* integer will fail, so we try *odd* factors of 63.

   Trying $x = \pm 3$, we have $27+27+9-63=0$. Bingo!
   The given equation factors to $\left(|x|^2+6|x|+21\right)\left(|x|-3\right)$ and, the quadratic factor, as the sum of 3 positive terms cannot be zero, so there are only 2 answers, $\underline{\pm 3}$.

2. $|2x-5| < A \Leftrightarrow \dfrac{5-A}{2} < x < \dfrac{5+A}{2}$
   A must be positive. Examining a few consecutive values of *A*, a pattern becomes clear.
   $A=1 \Rightarrow 2<x<3 \Rightarrow 0$ integer solutions
   $A=2 \Rightarrow 1.5<x<3.5 \Rightarrow 2$ integer solutions
   $A=3 \Rightarrow 1<x<4 \Rightarrow 2$ integer solutions
   $A=4 \Rightarrow .5<x<4.5 \Rightarrow 4$ integer solutions
   $A=5 \Rightarrow 0<x<5 \Rightarrow 4$ integer solutions
   $A=6 \Rightarrow -0.5<x<5.5 \Rightarrow 6$ integer solutions
   There is always an even number of integer solutions.
   If *A* is even, there are *A* solutions.
   If *A* is odd, there are $(A-1)$ solutions.
   Thus, *A* must be at least **2018**.

3. $8x+\dfrac{27}{x^2} > 4-\dfrac{6}{x}+\dfrac{9}{x^2} \Leftrightarrow \dfrac{8x^3+27}{x^2} > \dfrac{4x^2-6x+9}{x^2} \Rightarrow 8x^3+27 > 4x^2-6x+9 \; (x \neq 0)$
   Factoring the left side and transposing terms to the left side of the inequality,
   $(2x+3)(4x^2-6x+9)-(4x^2-6x+9) > 0 \Leftrightarrow (4x^2-6x+9)(2x+2) > 0$
   Examining the trinomial factor,
   $$4x^2-6x+9 = 4\left(x^2-\frac{3}{2}x+\boxed{\frac{9}{16}}\right)+9-\boxed{\frac{9}{4}} = 4\left(x-\frac{3}{4}\right)^2+\frac{27}{4}$$
   we see it is always positive. Thus, $2x+2>0 \Rightarrow x>-1$.
   But, since *x* cannot be 0, we have $\underline{\mathbf{-1<x<0,\ x>0}}$.

Detailed Solutions for GBML Meet 4 - JANUARY 2017

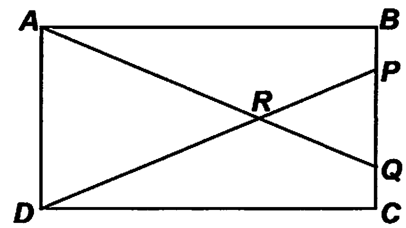
ROUND 3

1. Let the sides of ΔABC be denoted $2a$, $2b$, and $2c$, where $2c$ denotes the hypotenuse. Then:

   On the legs, we have $\dfrac{\pi a^2}{2} = 9\pi \Rightarrow a = 3\sqrt{2}$, $\dfrac{\pi b^2}{2} = 16\pi \Rightarrow b = 4\sqrt{2}$

   On the hypotenuse, we have $\dfrac{\pi c^2}{2} = 25\pi \Rightarrow c = 5\sqrt{2}$

   Thus, the area of ΔABC is $\frac{1}{2}(2a)(2b) = 2ab = 2\left(3\sqrt{2}\right)\left(4\sqrt{2}\right) = \underline{\mathbf{48}}$.



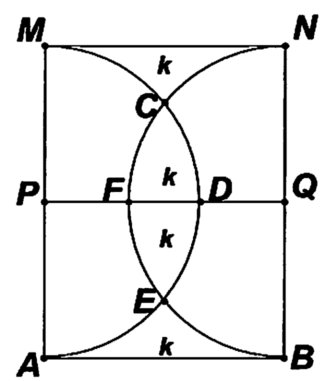
2. $\dfrac{BP}{PC} = \dfrac{3a}{7a} \Rightarrow BC = 10a$, $\dfrac{QC}{QB} = \dfrac{4b}{11b} \Leftrightarrow BC = 15b \quad \therefore 10a = 15b$

   Therefore, without loss of generality (WLOG), we take

   $BC = LCM(10,15) = 30 \Rightarrow a = 3, b = 2$

   $\Rightarrow AD = 30,\ BP = 9,\ CQ = 8,\ PQ = 13$

   Since $\Delta PQR \sim \Delta DAR$, the ratio of the areas is $13^2 : 30^2 = \underline{\mathbf{169:900}}$.



3. If x denotes the radius of the semi-circles, then the area of rectangle *AMNB* is $2x$. The area of *AMNB* equals the area of a circle of radius x minus the overlap of the two semi-circles (*CDEF*) plus the area of the regions inside the rectangle and outside both semi-circles (*MCN* and *AEB*). Thus,

   $2x = \pi x^2 - 2k + 2k \Rightarrow \text{area}(PMNQ) = x = \underline{\dfrac{2}{\pi}}$.

Detailed Solutions for GBML Meet 4 - JANUARY 2017

ROUND 4

1. $\left(1-\frac{a}{1}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{6}\right)=(1-a)\cdot\frac{1}{\not 2}\cdot\frac{\not 2}{\not 3}\cdot\frac{\not 3}{\not 4}\cdot\frac{\not 4}{\not 5}\cdot\frac{\not 5}{6}=\frac{1-a}{6}=-a$

$\Rightarrow 1-a=-6a\Rightarrow a=-\frac{1}{5}=-0.2$. Since $|a|<1$, the geometric series $1+a+a^2+a^3+a^4+\ldots$

converges to $\frac{a}{1-r}=\frac{1}{1-(-0.2)}=\frac{1}{1.2}=\underline{\frac{5}{6}}$.

2. If $N=1.\bar{1}_{(\text{base }9)}=\left(1+\frac{1}{9^1}+\frac{1}{9^2}+\frac{1}{9^3}+\ldots\right)$, then $9N=9+\left(1+\frac{1}{9^1}+\frac{1}{9^2}+\frac{1}{9^3}+\ldots\right)=9+N\Rightarrow N=\frac{9}{8}$

Thus, $1.6\bar{1}_{(\text{base }9)}=0.5_{(\text{base }9)}+1.\bar{1}_{(\text{base }9)}=\frac{5}{9}+\frac{9}{8}=\frac{40+81}{72}=\frac{121}{72}_{(\text{base }10)}$.

Converting numerator and denominator to base 7 integers,

$\begin{cases}121=98+21+2=2(7^2)+3(7)+2=232_{(7)}\\72=49+21+2=1(7^2)+3(7)+2=132_{(7)}\end{cases}$. Thus, $0.5\bar{4}_{(\text{base }9)}=\underline{\frac{121}{72}_{(base\,10)}}=\underline{\mathbf{\frac{232}{132}}_{(\text{base }7)}}$.

Note that $\frac{58}{33}_{(7)}$ is incorrect. Even/Odd are base 10 concepts and dividing by 4 is inappropriate, since the numerator is actually an *odd* number.

Alternately, note that $N=1+\frac{1}{9^1}+\frac{1}{9^2}+\frac{1}{9^3}+\ldots$ is an infinite geometric series with

$(a,r)=\left(1,\frac{1}{9}\right)$ and $S_\infty=\frac{a}{1-r}=\frac{1}{1-\frac{1}{9}}=\frac{9}{8}$.

3. $(j+k)^3=j^3+k^3+3\left(j^2k+jk^2\right)=100+3(300)=1000\Rightarrow j+k=10\Rightarrow j=10-k$.

[The other values of $(j+k)$ are complex, namely, $5\left(-1\pm i\sqrt{3}\right)$.]

Substituting, $j^3+(10-j)^3=100$

$\Rightarrow j^3+1000-300j+30j^2-j^3=100$

$\Rightarrow 30j^2-300j+900=0\Leftrightarrow j^2-10j+30=0$

$\Rightarrow j=\frac{10\pm\sqrt{100-120}}{2}=\frac{10\pm 2i\sqrt{5}}{2}=5\pm i\sqrt{5}\Rightarrow$

$(j,k)=\underline{\mathbf{\left(5+i\sqrt{5},5-i\sqrt{5}\right),\left(5-i\sqrt{5},5+i\sqrt{5}\right)}}$

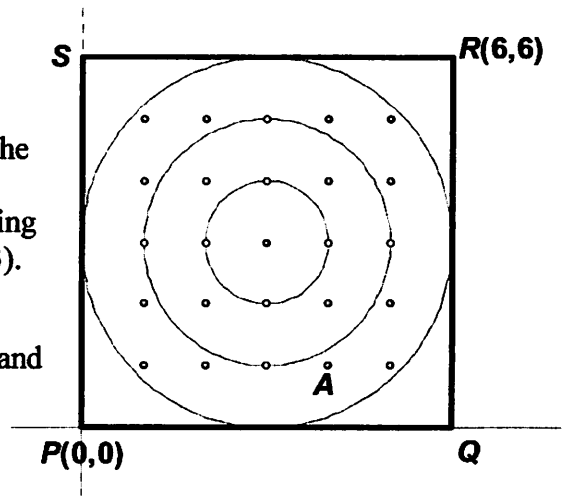
Check: $\left(5+i\sqrt{5}\right)^2\left(5-i\sqrt{5}\right)+\left(5+i\sqrt{5}\right)\left(5-i\sqrt{5}\right)^2$

$\Leftrightarrow\left(20+10i\sqrt{5}\right)\left(5-i\sqrt{5}\right)+\left(5+i\sqrt{5}\right)\left(20-10i\sqrt{5}\right)$

$\Leftrightarrow\left(100+50+30i\sqrt{5}\right)+\left(100+50-30i\sqrt{5}\right)=300$

Detailed Solutions for GBML Meet 4 - JANUARY 2017

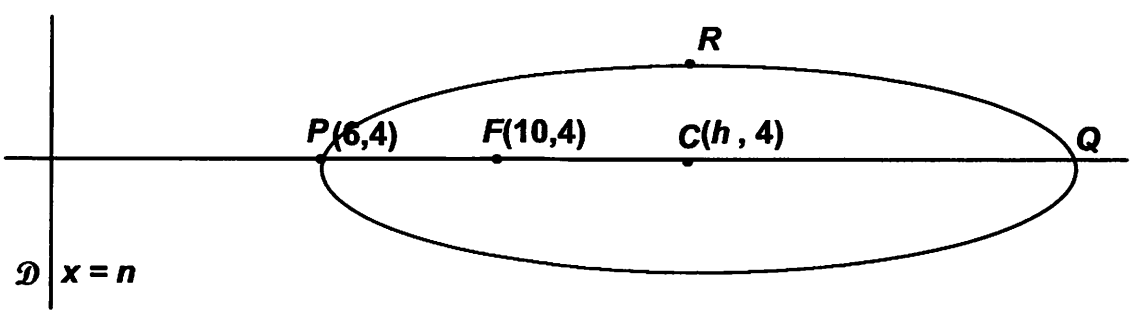
ROUND 5



1. The sketch at the right suggests that (5,5) is in ring A? If this is actually the case, it is the farthest point from the origin; otherwise, either (5,4) or (4,5) are tied for the farthest and multiple h-values are possible. Points in ring A are more than 2 units, but less than 3 units from (3, 3).
   $(3,3) \rightarrow (5,5) \Rightarrow \sqrt{2^2+2^2} = \sqrt{8}$ and $2 < \sqrt{8} < 3$,
   confirming that the required line passes through (2, 1) and (5, 5). The equation of this line is $(y-1) = \frac{4}{3}(x-2)$.
   Setting $y = 0$, $(x-2) = \frac{3}{4}(-1) \Rightarrow x = \underline{\frac{5}{4}}$.

2. For $y = 2$, $x^2 = 8 \Leftrightarrow x = \pm 2\sqrt{2} \Rightarrow RQ = 4\sqrt{2}$.
   For $y = k$, $ST = 8\sqrt{2} \Rightarrow x = \pm 4\sqrt{2} \Rightarrow x^2 = 32 \Rightarrow (y-1) = \frac{32}{8} = 4 \Rightarrow y = 5$.
   Thus, $k = \underline{5}$.

3. Let $C(h,4)$ be the center of the ellipse.



The eccentricity $e = \frac{PF}{P\mathfrak{D}} = \frac{c}{a} = \frac{h-10}{h-6} = \frac{1}{2} \Rightarrow \begin{cases} 2h-20 = h-6 \Rightarrow h = 14 \\ \frac{4}{6-n} = \frac{1}{2} \Rightarrow 6-n = 8 \Rightarrow n = -2 \end{cases}$

$\Rightarrow c = 14-10 = 4$, $a = 14-6 = 8$ and $a^2 = b^2 + c^2 \Rightarrow b^2 = 48 \Rightarrow b = 4\sqrt{3}$

Since the uppermost point is b units above the center, $(n, q_1, r_1, r_2) = \underline{\left(\mathbf{-2, 22, 14, 14+4\sqrt{3}}\right)}$.

Detailed Solutions for GBML Meet 4 - JANUARY 2017

TEAM ROUND

1. The semi-perimeter is 24.
   Let x and $27-x$ denote the lengths of the other two sides.
   Using Hero's formula, the area of the triangle is
   $3\sqrt{24\cdot 3\cdot(24-x)(24-(27-x))}=6\sqrt{2\cdot(24-x)(x-3)}$
   Clearly, $x\geq 4$ and the radicand must be a perfect square
   $4\Rightarrow 2\cdot 20\cdot 1 \quad 5\Rightarrow 2\cdot 19\cdot 2$
   $6\Rightarrow 2\cdot 18\cdot 3 \quad 7\Rightarrow 2\cdot 17\cdot 4$
   $8\Rightarrow 2\cdot 16\cdot 5 \quad 9\Rightarrow 2\cdot 15\cdot 6$
   $10\Rightarrow 2\cdot 14\cdot 7=14^2$
   Thus, a triangle with sides of lengths 10, 17 and 21 has an area of 84.
   The shortest side is **<u>10</u>**.
   Are there others?
   We need only check out 11-16-21, 12-15-21, 13-14-21. After these, the lengths of the unknown sides just reverse. All of these fail to produce an integer area, so the answer is unique.

2. Let $A=2n+1$ and $h=0.01$. Then $2n+1-\sqrt{4n^2+4n}<0.01 \Leftrightarrow A-\sqrt{A^2-1}<h$
   $\Leftrightarrow A<h+\sqrt{A^2-1}$ (Note: Both sides of the inequality are positive.)
   Squaring both sides, $A^2<h^2+2h\sqrt{A^2-1}+A^2-1$
   $\Leftrightarrow \dfrac{1-h^2}{2h}<\sqrt{A^2-1}$ (Again, both sides are positive.)
   Squaring both sides again, we have
   $$A^2>\left(\frac{1-h^2}{2h}\right)^2+1=\frac{1-2h^2+h^4+4h^2}{4h^2}=\frac{1+2h^2+h^4}{4h^2}=\frac{(1+h^2)^2}{(2h)^2}$$
   $$\Rightarrow A>\frac{1+h^2}{2h}=\frac{1}{2h}+\frac{h}{2}$$
   Substituting, $2n+1>\dfrac{1}{2\left(\frac{1}{100}\right)}+\dfrac{0.01}{2}=50+0.005\Rightarrow n_{\min}=\underline{\mathbf{25}}$.

Detailed Solutions for GBML Meet 4 - JANUARY 2017

TEAM ROUND - continued

3. The required area is

$$\frac{1}{2}\begin{vmatrix} 5 & -5 \\ 3t & -3 \\ -1 & -t \\ 5 & -5 \end{vmatrix} = \frac{1}{2}\left|\left(\underbrace{(5\cdot -3)+(3t\cdot -t)+(-1\cdot -5)}_{\text{down diagonal products}}\right)-\left(\underbrace{(5\cdot -t)+(-1\cdot -3)+(3t\cdot -5)}_{\text{up diagonal products}}\right)\right| = 50$$

$\Leftrightarrow \left|-15-3t^2+5+5t-3+15t\right| = 100$

$\Leftrightarrow \left|-3t^2+20t-13\right| = 100$

$\Leftrightarrow 3t^2-20t+13 = \pm 100$

Case 1: $3t^2-20t-87=0 \Leftrightarrow (t+3)(3t-29)=0 \Rightarrow t = \mathbf{-3, \underline{\frac{29}{3}}}$.

Case 2: $3t^2-20t+113=0$ Discriminant $= 20^2-4\cdot 3\cdot 113<0 \Rightarrow$ only imaginary roots

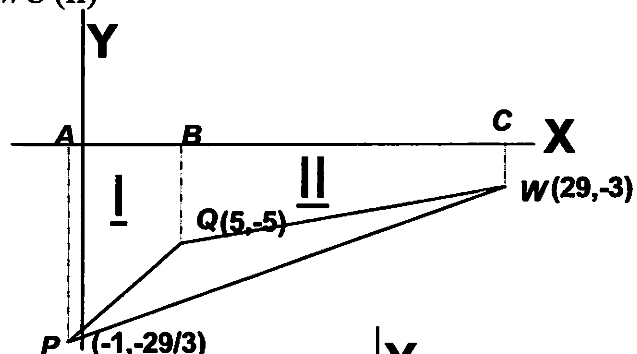
Alternate argument:
Drop perpendiculars to the x-axis from the vertices of ΔPQW.
The area of ΔPQW may be determined as the area of trapezoid $APWC$, minus the sum of the areas of trapezoids $APQB$ (I) and $BQWC$ (II)

For $x = \frac{29}{3}$:

$$\frac{1}{2}\left[30\left(3+\frac{29}{3}\right)-\left[24(3+5)+6\left(5+\frac{29}{3}\right)\right]\right]$$

$\Rightarrow (45+145)-(36+60)-(15+29)$

$\Rightarrow 190-96-44=50$



For $x=-3$, translate each point UP 5 units, so the triangle is ABOVE the x-axis.

$\text{Trap}(WABP)+\Delta PBQ-\Delta WAQ$

$\Rightarrow \frac{1}{2}\left[8(8+2)+6\cdot 8-2\cdot 14\right]=40+24-14=50$