GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 - JANUARY 2018

ROUND 1 - Volume and Surface Area of Solids

1. ______________________________

2. (____________ , ____________)

3. ______________________________

1. A right circular cone has a base with area 25π square inches and an altitude of 18 inches. A cross-section parallel to the base has an area of 9π square inches. Compute the distance between the base and this cross section.

2. Each of the edges of a regular pyramid with a square base has a length of 6 feet. This pyramid has volume of V cubic feet and a <u>total</u> surface area of A square feet. Compute the ordered pair (V, A)

3. The base of a tetrahedron is an equilateral triangle of side 4 inches. Its three lateral edges of have lengths 3 inches, 5 inches, and 5 inches. Compute the volume of this tetrahedron.

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ROUND 2 - Inequalities and Absolute Value

1. ______________________

2. ______________________

3. ______________________

Solve for x over the real numbers.

1. $\dfrac{x}{2x-1} - \dfrac{x+1}{2x+1} \le \dfrac{2x}{4x^2-1}$

2. $\dfrac{3-x}{x} + \dfrac{2}{|x|} \ge \dfrac{1}{2} - \dfrac{1}{|x|}$

3. $|3x-2| \le 7-x$ <u>and</u> $|9-2x| \ge x+7$

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 3 - Similar Polygons, Circles and Area

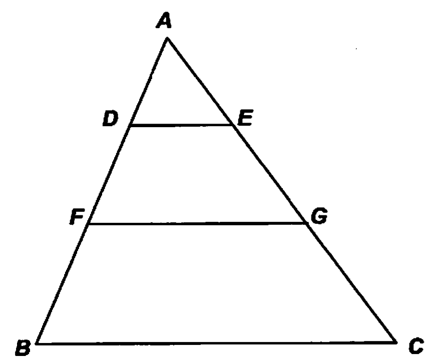
1. ____________ : ______________

2. ____________________________

3. ____________ : ______________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

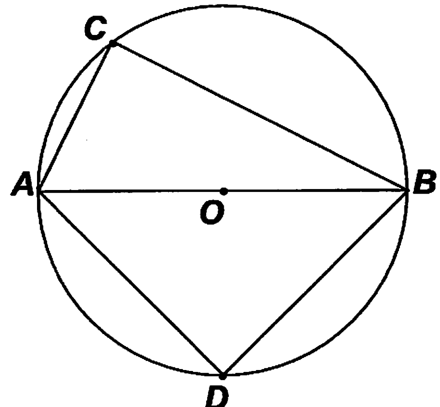
Let $\mathcal{A}(\ )$ denote the area of a region.



1. Given: ΔABC with $\overline{DE} \parallel \overline{FG} \parallel \overline{BC}$

   $AF : FB = 8 : 7$ and $AD : DB = 1 : 4$

   Compute $\mathcal{A}(\Delta AFG) : \mathcal{A}(DECB)$.

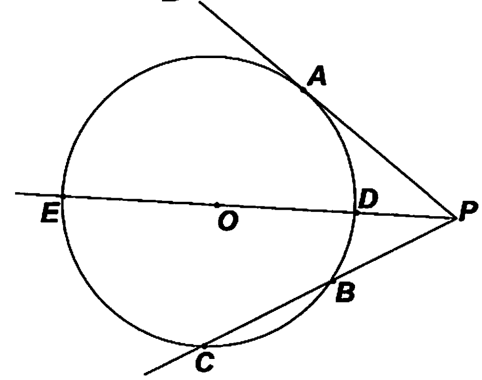


2. Given: Circle O, diameter $\overline{AB}$, $m\angle ABC = x°$,

   and (minor) arc $\widehat{AC} = \left(\frac{x}{2} + 45\right)^{\circ}$

   $AD = DB = \sqrt{3}$

   Compute $\mathcal{A}(\Delta ABC) - \mathcal{A}(\Delta ADB)$.



3. $\overleftrightarrow{PA}$ is tangent to circle O at point A.

   $\overleftrightarrow{PO}$ and $\overleftrightarrow{PC}$ are secants.

   $PD = 2x$, $EO = 3x$, $PB = x + 5$, where x is an integer.

   PA and BC are also integer lengths.

   For a <u>minimum</u> value of x, compute the ratio PA:BC.

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 4 - Sequences and Complex Numbers

1. ______________________

2. (____________ , ____________)

3. (_____ , _____ , _____ , _____)

1. Given the geometric sequence $1+i,\ 2i,\ 2i-2,\ -4,\ -4-4i, \ldots$
   If $t_1 = 1+i$ and $t_n = 256$, compute *n*.

2. Two numbers *x* and *y*, where $x < y$, differ by 15. Their arithmetic mean exceeds their geometric mean by $1\frac{1}{2}$. Compute the ordered pair (x, y).

3. Given: $x^3 = -8+8i$
   The three values of *x*, expressed in polar form (angles in degrees), which solve this equation are $r cis\theta_1$, $r cis\theta_2$, and $r cis\theta_3$, where $r > 0$ and $0° \le \theta_1 < \theta_2 < \theta_3 < 360°$.
   Compute the order quadruple $(r, \theta_1, \theta_2, \theta_3)$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 - JANUARY 2018

ROUND 5 - Conics

1. ______________________

2. ______________________

3. (________ , ________ , ________)

1. Given: $A(1,9)$, $B(3,7)$
   A line L_1 is perpendicular to $\overline{AB}$ at its midpoint M and crosses the x-axis at point K.
   Compute the length of segment $\overline{MK}$.

2. Given: A parabola defined by $y^2 - 8x - 4y - 12 = 0$ and a line defined by $x - y + 4 = 0$
   Compute the length of the chord determined by the intersection of the line and the parabola.

3. Given: a conic with equation $4x^2 - 25y^2 = 8x + 100y + 196$
   Let $C(h,k)$ be the center of this conic and d be the length of the segment connecting a focus and the <u>closer</u> vertex. Compute the ordered triple (h,k,d).

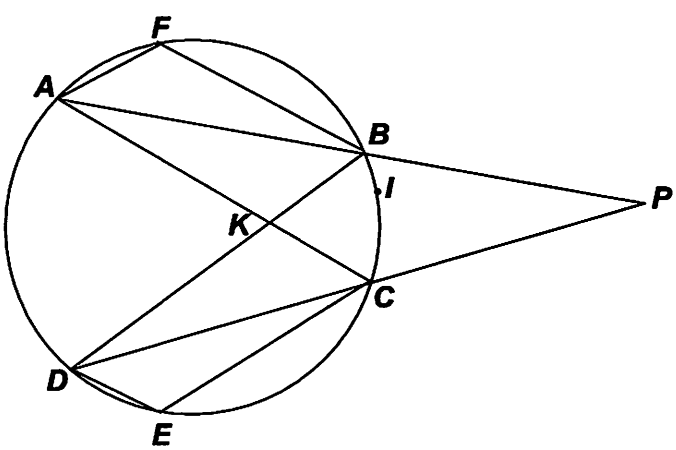
GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 - JANUARY 2018

TEAM ROUND

3 pts.	1.	__________ : __________
3 pts.	2.	____________________
4 pts.	3.	____________________

1. Given:
   $m\angle AFB = 120°$, $m\angle DEC = 110°$, $\overset{\frown}{BIC} = 40°$
   Compute the ratio $m\angle AKD : m\angle P$.



2. Solve for x over the real numbers: $\dfrac{|3x-4|}{|x+2|} \geq \dfrac{3}{2}$.

3. The numbers n_1, n_2, and n_3 form a geometric progression; their sum is 78.
   The third number, n_3, is 2.25 times the sum of the other two. Compute <u>all</u> possible ordered triples (n_1, n_2, n_3).

GREATER BOSTON MATHEMATICS LEAGUE

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Answer Sheet

Round 1

1. 7.2
2. $\left(36\sqrt{2},\ 36\left(\sqrt{3}+1\right)\right)$
3. $4\sqrt{3}$

Round 2

1. $-\frac{1}{2}<x<\frac{1}{2},\ x>\frac{1}{2}$ (or equivalent)
2. $0<x\leq 4$
3. $-\frac{5}{2}\leq x\leq\frac{2}{3}$

Round 3

1. 8:27
2. $\frac{3\left(\sqrt{3}-2\right)}{4}\left[\text{or } \frac{3\sqrt{3}-6}{4}\right]$
3. 6:5

Round 4

1. 16
2. $(12,27)$
3. $\left(2\sqrt[6]{2},45,165,285\right)$
   Note: $2\sqrt[6]{2}$ may be expressed as $2^{7/6}$.

Round 5

1. $8\sqrt{2}$
2. $8\sqrt{2}$
3. $\left(1,-2,\sqrt{29}-5\right)$

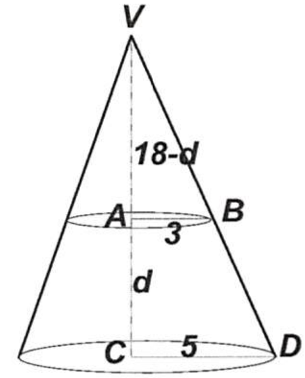
Team Round

1. 5:1 (3 pts)
2. $\begin{cases} x<-2 \\ -2<x\leq\frac{2}{9} \\ x\geq\frac{14}{3} \end{cases}$ (3 pts)
   or equivalent
3. $(6,18,54)$, $(96,-72,54)$ (4 pts)

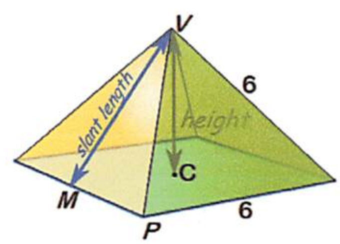
Detailed Solutions for GBML Meet 4 - JANUARY 2018

ROUND 1

1. Since $\Delta VAB \sim \Delta VCD$
$\frac{3}{5} = \frac{18-d}{18} \Rightarrow 54 = 90 - 5d \Rightarrow 5d = 36 \Rightarrow d = \underline{\mathbf{7.2}}$.

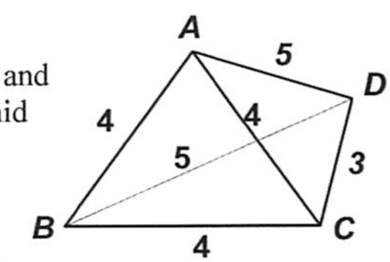


2. Let V be theapex of the regular pyramid, C be the center of the square base, and P be one of the vertices of the square base. B, the area of the base, = 36.
Then: $VP = 6$ (a lateral edge),
$CP = \frac{1}{2}(6\sqrt{2}) = 3\sqrt{2}$ (half a diagonal of the base),
$VC = h_1$ (the altitude of the pyramid)
Since VPC is a right triangle, we have
$h_1^2 + (3\sqrt{2})^2 = 6^2 \Rightarrow h_1 = 3\sqrt{2}$



Let M be the midpoint of a side of the square base. $CM = 3$, $VM = h_2$ (an altitude of a face)
Since VCM is also a right triangle, we have $(3\sqrt{2})^2 + 3^2 = h_2^2 \Rightarrow h_2 = \sqrt{27} = 3\sqrt{3}$
Thus, the volume is $\frac{1}{3}h_1 B = \frac{1}{3} \cdot 3\sqrt{2} \cdot 36 = 36\sqrt{2}$.
The total surface area is 4 (area of a face) + B = $4\left(\frac{1}{2} \cdot 6 \cdot 3\sqrt{3}\right) + 36 = 36(\sqrt{3}+1)$
$\Rightarrow (V, A) = \underline{\mathbf{(36\sqrt{2},\ 36(\sqrt{3}+1))}}$

3. In the spatial diagram at the right, ΔABC is in the foreground and point D is in the background. $ABCD$ is a tetrahedron, a pyramid with a triangular base.
With sides of 3-4-5, both ΔACD and ΔBCD must be right triangles. In each case, the right angle is at point C. Thus, $\overline{DC}$ is perpendicular to the plane containing $\overline{AC}$ and $\overline{BC}$, i.e., perpendicular to the face ΔABC. Therefore, we can use equilateral triangle ABC as the base of the tetrahedron and $\overline{DC}$ as the corresponding altitude.



Finally, $V = \frac{1}{3} \cdot 3 \cdot \frac{4^2\sqrt{3}}{4} = \underline{\mathbf{4\sqrt{3}}}$.

Detailed Solutions for GBML Meet 4 - JANUARY 2018

ROUND 2

1. Given: $\frac{x}{2x-1} - \frac{x+1}{2x+1} \le \frac{2x}{4x^2-1}$ Clearly, $x \ne \pm\frac{1}{2}$

$\Leftrightarrow \frac{x(2x+1)-(x+1)(2x-1)}{(2x-1)(2x+1)} - \frac{2x}{4x^2-1} \le 0$

$\Leftrightarrow \frac{(2x^2+x-2x^2+x-2x+1)-2x}{4x^2-1} \le 0$

$\Leftrightarrow \frac{-2x+1}{4x^2-1} \le 0 \Leftrightarrow \frac{2x-1}{4x^2-1} \ge 0 \Leftrightarrow \frac{1}{2x+1} \ge 0 \Leftrightarrow 2x+1 \ge 0 \Leftrightarrow x \ge -\frac{1}{2}$

But we must exclude both $x = -\frac{1}{2}$ and $x = \frac{1}{2}$!! Thus, the solution is an open-ended ray with a single point missing, namely, $\underline{\mathbf{-\frac{1}{2} < x < \frac{1}{2},\ x > \frac{1}{2}}}$ (or equivalent).

2. Given: $\frac{3-x}{x} + \frac{2}{|x|} \ge \frac{1}{2} - \frac{1}{|x|}$

Case 1: $x > 0$

$\Leftrightarrow \frac{3-x}{x} + \frac{2}{x} \ge \frac{1}{2} - \frac{1}{x}$

Multiplying by $2x$ (which is positive quantity), $6-2x+4 \ge x-2 \Leftrightarrow 12 \ge 3x \Leftrightarrow 4 \ge x$
Considering the domain of definition restriction, we have $0 < x \le 4$.
Case 2: $x < 0$

$\frac{3-x}{x} + \frac{2}{-x} \ge \frac{1}{2} - \frac{1}{-x} \Leftrightarrow \frac{3-x}{x} - \frac{2}{x} \ge \frac{1}{2} + \frac{1}{x}$

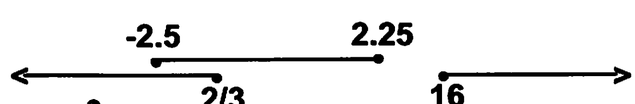
Multiplying by $2x$ (which is negative quantity), $6-2x-4 \le x+2 \Leftrightarrow 0 \le 3x \Leftrightarrow x \ge 0$
Considering the domain of definition restriction, all of the values are extraneous.
Thus, the solution is $\underline{\mathbf{0 < x \le 4}}$.

3. $|3x-2| \le 7-x \Leftrightarrow x-7 \le 3x-2 \le 7-x$

$\Leftrightarrow 2x \ge -5$ <u>and</u> $4x \le 9 \Leftrightarrow -\frac{5}{2} \le x \le \frac{9}{4}$ (a segment with closed endpoints)

$|9-2x| \ge x+7 \Leftrightarrow 9-2x \le -x-7$ or $9-2x \ge x+7$

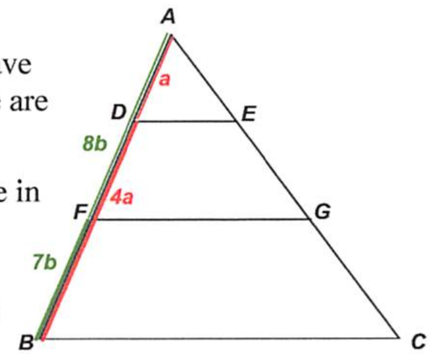
$\Leftrightarrow x \ge 16$ <u>or</u> $x \le \frac{2}{3}$ (two rays with closed endpoints)



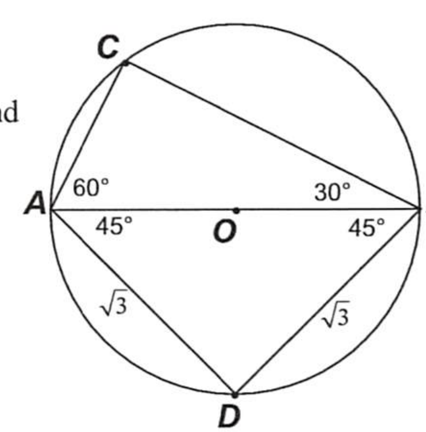
Taking the intersection, we have $\underline{\mathbf{-\frac{5}{2} \le x \le \frac{2}{3}}}$.

Detailed Solutions for GBML Meet 4 - JANUARY 2018

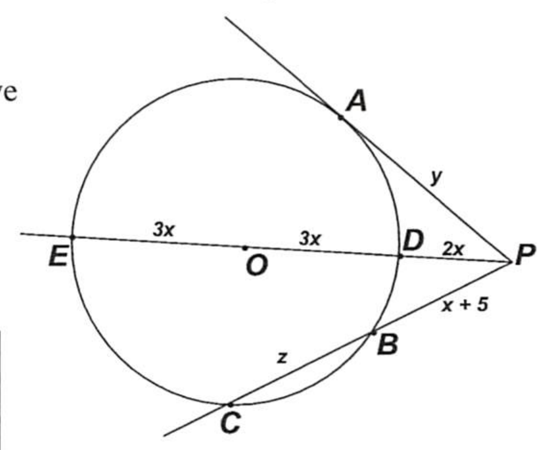
ROUND 3



1. For side $\overline{AB}$, using the given ratios of 8:7 and1 : 4, we have $AB = 5a = 15b$. Thus, $a = 3b$, and, *arbitrarily*, (since we are only interested in ratios), we let $(a,b) = (3,1)$. Since $\Delta ADE \sim \Delta AFG \sim \Delta ABC$ and their corresponding sides are in a known ratio, namely ($AD : AF : AB = 3:8:15$), their areas must be in a $9:64:225$ ratio. Subtracting the areas of the overlapping regions,
   $\mathcal{A}(\Delta ADE) : \mathcal{A}(DEFG) : \mathcal{A}(FGCD) = 9:55:161$
   Thus, $\mathcal{A}(\Delta AFG) : \mathcal{A}(DECB) = 64:(225-9) = 64:216 = \underline{\mathbf{8:27}}$.



2. $\widehat{AC} = 2x = \frac{x}{2} + 45° \Rightarrow x = 30°$. Since angles inscribed in a semicircle are always right angles, we have $m\angle CAB = 60°$ and $m\angle DAB = \angle DBA = 45°$.
   Thus, $AB = \sqrt{6} \Rightarrow AC = \frac{1}{2}\sqrt{6},\ BC = \left(\frac{1}{2}\sqrt{6}\right)\sqrt{3} = \frac{3}{2}\sqrt{2}$.
   $$\mathcal{A}(\Delta ABC) - \mathcal{A}(\Delta ADB) = \frac{1}{2}\left(\frac{1}{2}\sqrt{6}\right)\left(\frac{3}{2}\sqrt{2}\right) - \frac{1}{2}\left(\sqrt{3}\right)^2$$
   $$= \frac{3}{8}\sqrt{12} - \frac{3}{2} = \frac{3\sqrt{3}}{4} - \frac{6}{4} = \underline{\mathbf{\frac{3\left(\sqrt{3}-2\right)}{4}}} \text{ or } \underline{\mathbf{\frac{3\sqrt{3}-6}{4}}}.$$
   Note: We knew the difference must be negative, because the isosceles triangle inscribed in a semicircle always has the maximum area.



3. For tangents and secants from a common point, we have
   $PA^2 = PD(PE) = PB(PC) \Rightarrow$
   $y^2 = (2x)(8x) = 16x^2 = (x+5)(x+5+z)$
   We must find the minimum integer x for which both y and z are also integers.

x	$PD(PE)$	y	$PB(PC)$	Verdict
1	2(8) = 16	4	6(PC) = 16	Rejected
2	4(16) = 64	8	7(PC) = 64	Rejected
3	6(24) = 144	12	8(PC) = 144	$PC = 18 \Rightarrow z = 10$

Thus,

$PA : BC = y : z = 12:10 = \underline{\mathbf{6:5}}$.

Detailed Solutions for GBML Meet 4 - JANUARY 2018

ROUND 4

1. $1+i,\ 2i,\ 2i-2,\ \mathbf{-4},\ -4-4i, \ldots$

   The common multiplier is $r=\dfrac{2i}{1+i}\cdot\dfrac{1-i}{1-i}=\dfrac{2i+2}{1+1}=1+i$.

   Note that the fifth term is -4 times the first term. Each term in the second group of four terms is likewise -4 times the corresponding term in the first group.

   Predictably, $t_4=-4,\ t_8=16,\ t_{12}=-64,\ t_{16}=256$.

   Thus, $n=\mathbf{16}$.

2. $AM=GM+\dfrac{3}{2}\Leftrightarrow\dfrac{x+(x+15)}{2}=\sqrt{x(x+15)}+1.5$

   $\Leftrightarrow 2x+15=2\sqrt{x(x+15)}+3$

   $\Leftrightarrow x+6=\sqrt{x(x+15)}$

   Squaring both sides, $x^2+12x+36=x^2+15x\Rightarrow x=12$

   Thus, $(x,y)=\mathbf{(12,27)}$.

3. Converting $-8+8i$ to $rcis\theta$ (polar form):

   $r^2=x^2+y^2\Rightarrow r^2=(-8)^2+(8)^2=128\Rightarrow r=2^{\frac{7}{2}}$

   $\tan\theta=\dfrac{y}{x}=\dfrac{8}{-8}=-1\Rightarrow\theta=135°,$ ~~315°~~ (since the original point is in quadrant 2)

   Let $x=rcis\theta$. Then: $x^3=(rcis\theta)^3=2^{\frac{7}{2}}cis135°$

   By DeMoivre's Theorem, $\begin{cases} r^3=2^{\frac{7}{2}} \\ 3\theta=135°+(360°)n \end{cases}\Rightarrow\begin{cases} r=2^{\frac{7}{6}}=2\sqrt[6]{2} \\ \theta=45°+(120°)n \end{cases}$

   $\Rightarrow x=2\sqrt[6]{2}cis\theta$, where $\theta=45°, 165°, 285°$. Thus, $(r,\theta_1,\theta_2,\theta_3)=\mathbf{(2\sqrt[6]{2}, 45, 165, 285)}$

   FYI: Check for $x=2\sqrt[6]{2}cis45°\Rightarrow\left(2\sqrt[6]{2}\right)^3 cis135°=8\sqrt{2}\left(-\dfrac{\sqrt{2}}{2}+i\cdot\dfrac{\sqrt{2}}{2}\right)=-8+8i$.

Detailed Solutions for GBML Meet 4 - JANUARY 2018

ROUND 5

1. $A(1,9)$, $B(3,7) \Rightarrow$ midpoint $\left(\frac{1+3}{2}, \frac{9+7}{2}\right) \Rightarrow M(2,8)$

   slope $m = \frac{7-9}{3-1} = -1 \Rightarrow m_{\perp} = +1$ (negative reciprocal)

   Thus, the equation of L_1 is $(y-8) = 1(x-2) \Leftrightarrow y = x+6$.

   $\Rightarrow$*x*-intercept @ $K(-6,0)$

   The distance $MK = \sqrt{(2-{}^{-}6)^2 + (8-0)^2} = \sqrt{2 \cdot 8^2} = \underline{\mathbf{8\sqrt{2}}}$.

2. $x - y + 4 = 0 \Leftrightarrow y = x + 4$

   Substituting in the equation for the parabola,

   $y^2 - 8x - 4y - 12 = 0 \Rightarrow (x+4)^2 - 8x - 4(x+4) - 12 = x^2 - 4x - 12 = 0 \Leftrightarrow (x-6)(x+2) = 0$

   $\Rightarrow x = 6, -2$

   $\Rightarrow$ Endpoints of the chord: $(6,10), (-2,2) \Rightarrow$ length of chord: $\underline{\mathbf{8\sqrt{2}}}$

3. $(4x^2 - 8x) - (25y^2 + 100y) = 196$

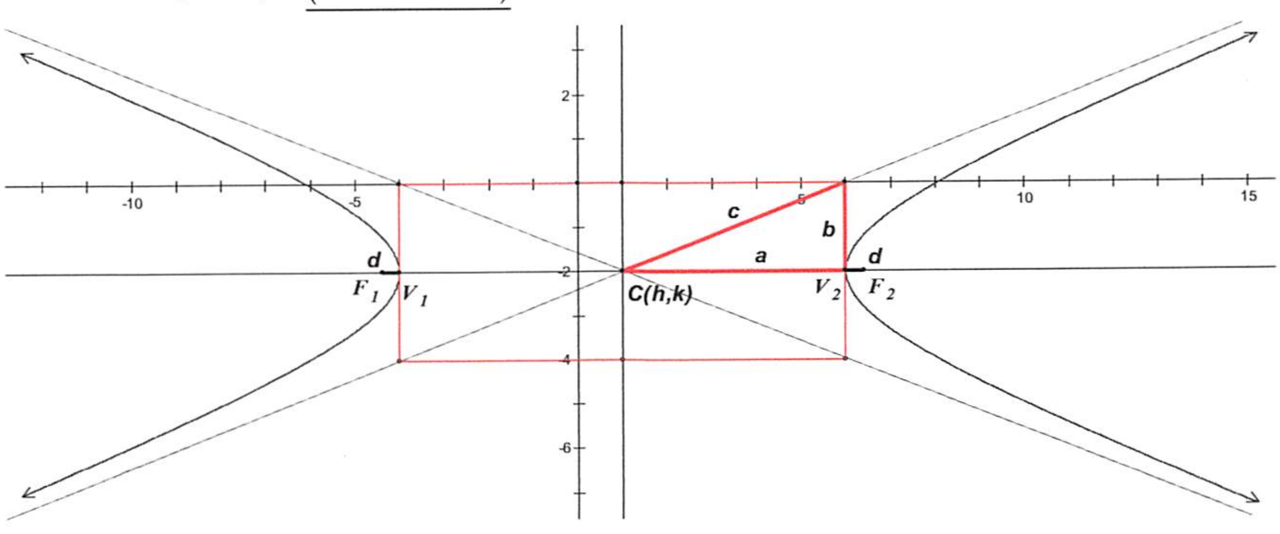
   $\Rightarrow 4(x^2 - 2x + 1) - 25(y^2 + 4y + 4) = 196 + 4 - 100 = 100$

   $\Rightarrow \frac{(x-1)^2}{25} - \frac{(y+2)^2}{4} = 1$

   A hyperbola with center at $(1,-2)$, horizontal major axis, $a = 5$, $b = 2$

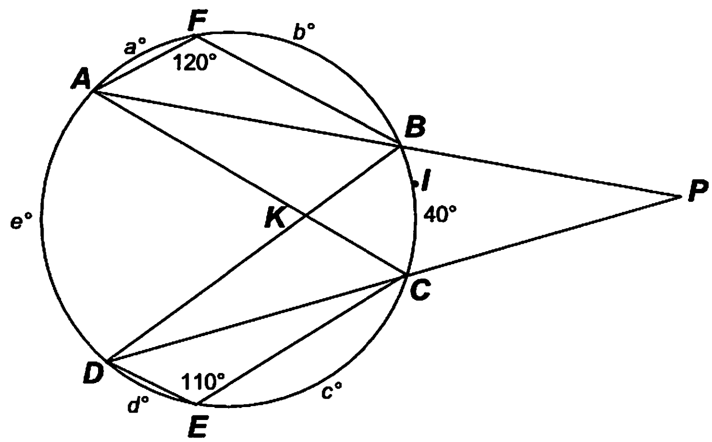
   Since $a^2 + b^2 = c^2$, we have $c^2 = 25 + 4 = 29$ and the required distance is $d = \sqrt{29} - 5$

   .Thus, $(h,k,d) = \underline{\left(\mathbf{1, -2, \sqrt{29} - 5}\right)}$.



Detailed Solutions for GBML Meet 4 - JANUARY 2018

TEAM ROUND



1. The intercepted arc of $\angle AFB$
$40+c+d+e=2(120°)$
$\Leftrightarrow \boxed{c+d+e=200}$
The intercepted arc of $\angle DEC=$
$e+a+b+40=2(110°)$
$\Leftrightarrow \boxed{a+b+e=180}$
Adding, $a+b+c+d+2e=380°$
However,
$a+b+c+d+e=360°-40°=320°$
$\Rightarrow e=60°$.
As an angle formed by two chords, $m\angle AKD=\dfrac{60+40}{2}=50°$. As an angle formed by two secants, $m\angle P=\dfrac{60-40}{2}=10° \Rightarrow m\angle AKD : m\angle P = \underline{\mathbf{5:1}}$.

2. Since each term in the quotients is positive, $\dfrac{|3x-4|}{|x+2|}\geq\dfrac{3}{2}\Leftrightarrow 2|3x-4|\geq 3|x+2|$
Again, since each product is positive, squaring both sides preserves the inequality.
$4(9x^2-24x+16)\geq 9(x^2+4x+4)\Leftrightarrow 27x^2-132x+28=(9x-2)(3x-14)\geq 0$
This product is nonnegative in the intervals outside the critical values of each factor, namely, over $x\leq\dfrac{2}{9}$ or $x\geq\dfrac{14}{3}$. However, $x=-2$ must be excluded!
Thus, the solution set is $x<-2,\ -2<x\leq\dfrac{2}{9},\ x\geq\dfrac{14}{3}$ (or equivalent).

3. Let $(t_1, t_2, t_3)=(a, ar, ar^2)$, where a denotes the first term and r is the common multiplier.
$$\begin{cases}(1)\ a+ar+ar^2=78\\ (2)\ ar^2=\dfrac{9}{4}(a+ar)\end{cases}$$
$(2)\Rightarrow r^2=\dfrac{9}{4}(1+r)\Leftrightarrow 4r^2-9r-9=(4r+3)(r-3)=0\Rightarrow r=-\dfrac{3}{4}, 3$
$r=-\dfrac{3}{4}\Rightarrow a-\dfrac{3}{4}a+\dfrac{9}{16}a=78\Rightarrow 16a-12a+9a=78\cdot16\Rightarrow a=\dfrac{78\cdot16}{13}=96\Rightarrow \underline{\mathbf{(96,-72,54)}}$.
$r=3\Rightarrow a+3a+9a=78\Rightarrow a=6\Rightarrow \underline{\mathbf{(6,18,54)}}$.