GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2015

ROUND 1 – Arithmetic: Open

1. ________________________

2. ________________________

3. ________________________

1. Let $K = 100^{15} - 15$ be expanded as a base 10 numeral.
   Compute the sum of all the digits in K.

2. How many natural numbers would be three-digit numbers when written in base 6 and base 8, or three-digit numbers when written in base 8 and base 20?

3. Find all values of A so that the seven-digit decimal number $1A45A32$ will be divisible by 24.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2015

ROUND 2 – Algebra 1: Open

1. ______________________

2. ______________________

3. ______________________

1. If x and y are integers, solve for <u>all</u> ordered pairs (x, y) for which $2^{2x} - 3^{2y} = 55$.

2. Simplify completely. $\left(4x^{-1} - x^{-2} - 5x^{-3}\right) \div \left(25x^{-3} - 16x^{-1}\right)$

3. Abe, Beau and Carlos working together, each at his own constant rate, can complete a job in 6 days. Abe and Beau working together can complete the same job in 7 days. If Carlos and Beau each work alone, Carlos would take three times as long as Beau would to complete this job. Honestly, how many days would it take Abe, working alone, to complete this job?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2015

ROUND 3 – Geometry: Open

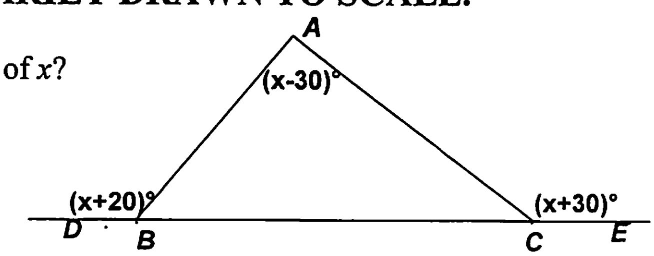
1. ______________________

2. (________ , ________ , ________)

3. ______________________

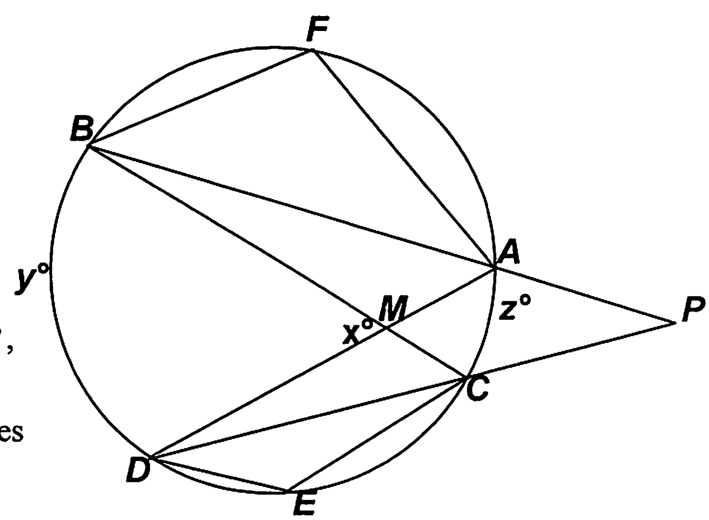
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. In the diagram at the right, what is the value of x? (D, B, C, and E are collinear.)

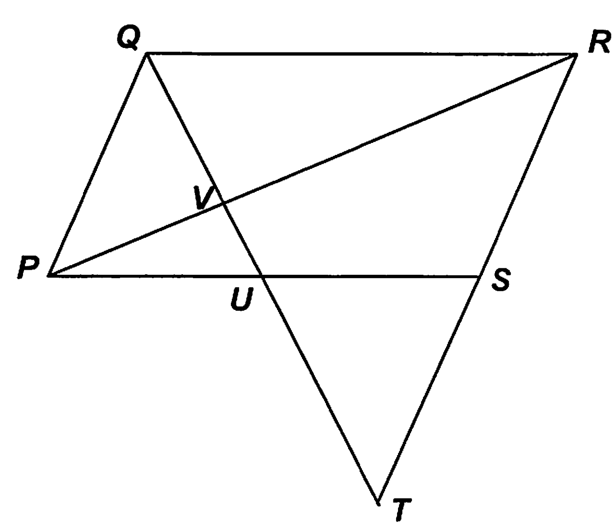


2. Given: $\overline{PAB}$ and $\overline{PCD}$ are secant lines

   $m\angle DEC = 133°$, $m\angle BFA = 115°$, $m\angle P = 42°$, $m\angle BMD = x°$
   Minor arcs $\overset{\frown}{BD}$ and $\overset{\frown}{AC}$ have degree measures indicated in the diagram.
   Compute the ordered triple (x, y, z).



3. $PQRS$ is a parallelogram.
   $\overrightarrow{QT}$ intersects $\overline{PR}$ at V, $\overline{PS}$ at U, $\overline{RS}$ at T.
   If $QU = 3$ and $QT = 6$, compute QV.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2015

ROUND 4 – Algebra 2: Open

1. (________ , ________ , ________)

2. ______________________________

3. ______________________________

1. The solution set for $\frac{x-1}{3x}-\frac{x}{x-1}+\frac{1}{x^2-x}<0$ is $\{x \mid x<A \text{ or } x>B \text{ and } x\neq C\}$.
   Determine the ordered triple (A,B,C).

2. The sum of an infinite geometric progression is 6 and the sum of the first 2 terms is 4.5 .
   Compute all possible values of the first term.

3. Given the equation $3x^2=J-Kx$.
   Compute all ordered pairs (K,J) which satisfy the following conditions:
   - One more than the sum of the roots is equal to the additive inverse of the product of the roots.
   - The product of the reciprocal of the roots is equal to $2\frac{1}{2}$ more than the sum of the reciprocals of the roots.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2015

ROUND 5 – Pre-Calculus: Open

1. ______________________

2. ______________________

3. ______________________

1. $J = \sum_{k=1}^{k=19} (-1)^k (k)(i^{2k})$, where $i = \sqrt{-1}$. Compute *J*.

2. Given: $0 \le \theta < 2\pi$
Compute the value(s) of θ for which the following statement is true:

$$\log\left(\cos\frac{5\pi}{3}\right) + \log\left(\cot\frac{7\pi}{6}\right) = \log(\tan\theta) + \log(\cos\theta)$$

3. Compute the value(s) of $\sin 2\theta$, where $0 \le \theta < 2\pi$ satisfies the following statement:

$$\sin\left(Tan^{-1}\left(-\frac{\sqrt{5}}{2}\right)\right) \cdot \cos\left(Sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right) = \cos\theta$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2015

TEAM ROUND

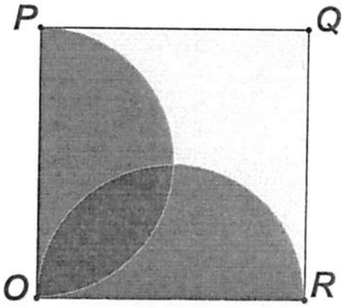
3 pts. 1. ____________________

3 pts. 2. (__________ , __________)

4 pts. 3. ____________________

1. Compute the value(s) of x for which $\left(\frac{2}{3x}+\frac{1}{8}\right)=\frac{3x}{2}+8$.

2. $OPQR$ is a square. Semi-circles are drawn on $\overline{OP}$ and $\overline{OR}$. The ratio of the area of the region inside the square and outside both semi-circles to the region common to both semi-circles is $\frac{A-\pi}{\pi-B}$. Compute the ordered pair (A, B).



3. How many district three-letter arrangements are possible if one letter is chosen from *SAN* and two letters are chosen from *ANTONIO*?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2015

Answer Sheet

Round 1

1. 265
2. 264
3. 0, 6

Round 2

1. $(3,1)$
2. $-\dfrac{x+1}{4x+5}$
3. 14

Round 3

1. 100
2. (68, 110, 26)
3. 2

Round 4

1. $(-2,0,1)$
2. 3, 9
3. $(7,-4)$

Round 5

1. 190
2. $\dfrac{\pi}{3}, \dfrac{2\pi}{3}$
3. $\pm\dfrac{4\sqrt{5}}{9}$

Team Round

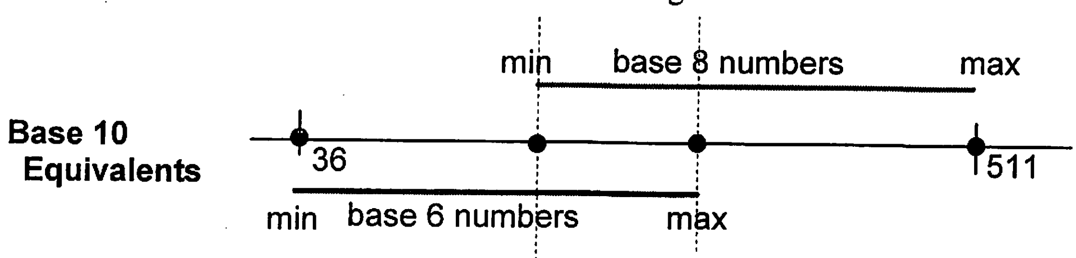
1. $\dfrac{1}{12}, -\dfrac{16}{3}$ (3 pts)
2. $(6,2)$ (3 pts)
3. 151 (4 pts)

Detailed Solutions for GBML Meet 5 – February 2015

ROUND 1

1. $100^2 = 10000$ - 4 zeros, 5 digits
   Subtracting 1 would produce 4 digits all 9s.
   $100^{15} = \underbrace{10......0}_{31\text{ digits}}$
   $100^{15} - 15 = 9.......985$ - 30 digits, including 28 9s and producing a digit sum of
   $28 \cdot 9 + (8+5) = 252 + 13 = 265$

2. A 3-digit number in base 6 is not necessarily a 3-digit number in base 8 ($100_6 = 36_{10} = 44_8$)
   A 3-digit number in base 8 is not necessarily a 3-digit number in base 6 ($777_8 = (1000_8 - 1) = 511_{10}$)
   The largest 3-digit number is base 6 is $555_6 = 1000_6 - 1 = 215_{10}$
   FYI: 511_{10} is a 4-digit number in base 6, namely, 2211_6
   In fact, we must take the intersection of these two ranges.



From the diagram it is clear that the numbers representable as 3-digit in both bases run from the smallest number in the larger base to the largest number in the smaller base.

Applying this to both cases:
Base 6 and 8: 100_8 to $555_6 \Leftrightarrow 8^2$ through $6^3 - 1 \Leftrightarrow 64$ to 215 inclusive, 152 values
Base 8 and 20: 100_{20} to $777_8 \Leftrightarrow 20^2$ through $8^3 - 1 \Leftrightarrow 400$ to 511 inclusive, 112 values
There is no overlap between these cases so, we have $152 + 112 = \underline{\mathbf{264}}$.

3. A number is divisible by 24 if and only if it is divisible by both 3 and 8.
   Since divisibility by 3 requires that the sum of the digits be divisible by 3, we require that $2A + 15$ be a multiple of 3. This occurs only for $A = 0, 3, 6$, or 9.
   Since divisibility by 8 is insured if the 3-digit number formed by the rightmost 3 digits is divisible by 8, we look at 032, 332, 632 and 932. Since 032 and 632 are multiples of 8, the required A-values are **0** and **6**.

Detailed Solutions for GBML Meet 5 – February 2015

ROUND 2

1. Since we require that the difference between a power of 2 and a power of 3 differ by 55, we need to consider lists of these numbers.
   Powers of 2: 1 2 4 8 16 32 64 128 256 512 1024 …
   Powers of 3: 1 3 9 27 81 243 729 …
   Looking for a number in the top row that is 55 more than a number in the second row.
   32 – 1 (too small), 64 – 1 (too large) - the spread is 32
   32 – 3 (too small), 64 – 3 (too large) - the spread is still 32
   64 – 9 (Bingo!)
   64 – 27 (too small), 128 – 27 (too large) – the spread is 64
   128 – 81 (too small), 256 – 81 (too large) – the spread is 128
   Since the spread is growing, there can be no additional pairs.
   This $2^{2x}=64$ and $3^{2y}=9 \Rightarrow (x,y)=\underline{\mathbf{(3,1)}}$.

   Alternate Solution (Difference of perfect squares):
   $2^{2x}-3^{2y}=55 \Leftrightarrow (2^x-3^y)(2^x+3^y)=5(11)$ This makes the solution $(x,y)=(3,1)$ obvious.
   How would you argue that this solution is unique?

2. $(4x^{-1}-x^{-2}-5x^{-3})\div(25x^{-3}-16x^{-1})=\left(\frac{4}{x}-\frac{1}{x^2}-\frac{5}{x^3}\right)\div\left(\frac{25}{x^3}-\frac{16}{x}\right)=\left(\frac{4x^2-x-5}{x^3}\right)\left(\frac{x^3}{25-16x^2}\right)$

   $=\frac{\cancel{(4x-5)}^{-1}(x+1)}{\cancel{x^3}}\cdot\frac{\cancel{x^3}}{(5+4x)\cancel{(5-4x)}^{1}}=\underline{-\frac{\mathbf{x+1}}{\mathbf{4x+5}}}$

3. Let Abe, Beau and Carlos complete the job in *a*, *b*, *c* hours, each working alone. Then:

   $$\begin{cases}(1)\ \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)6=1\\ (2)\ \left(\frac{1}{a}+\frac{1}{b}\right)7=1\\ (3)\ c=3b\end{cases}$$

   Using (2) and substituting in (1), we have

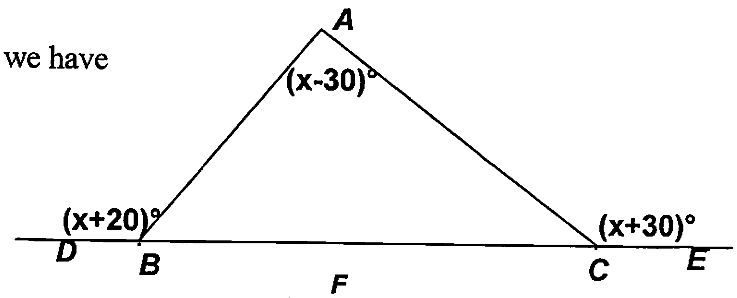
   $\left(\frac{1}{7}+\frac{1}{c}\right)6=1\Rightarrow\frac{6}{c}=\frac{1}{7}\Rightarrow c=42,\ b=14$.

   Substituting in (2), $\left(\frac{1}{a}+\frac{1}{14}\right)7=1\Rightarrow\frac{7}{a}=\frac{1}{2}\Rightarrow a=\underline{\mathbf{14}}$

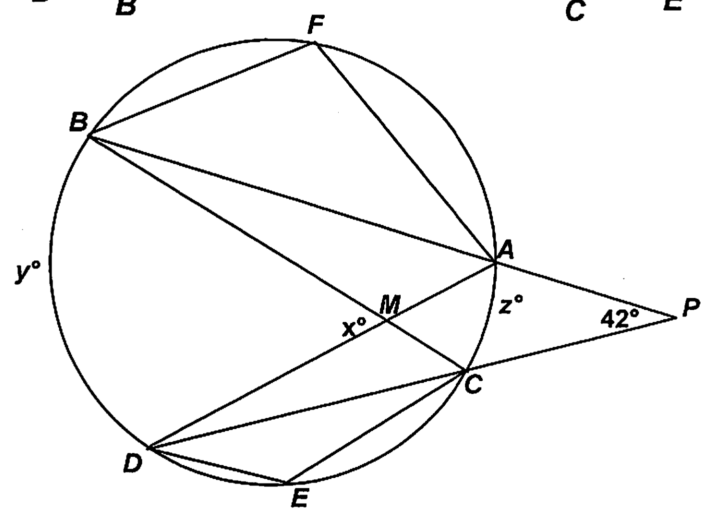
Detailed Solutions for GBML Meet 5 – February 2015

ROUND 3

1. $m\angle ABC = 180° - (x+20°) = 160° - x$
   Since $\angle ACE$ is an exterior angle of ΔABC, we have
   $x + 30° = (x - 30°) + (160° - x) = 130°$
   $\Rightarrow x = \underline{\mathbf{100}}$.

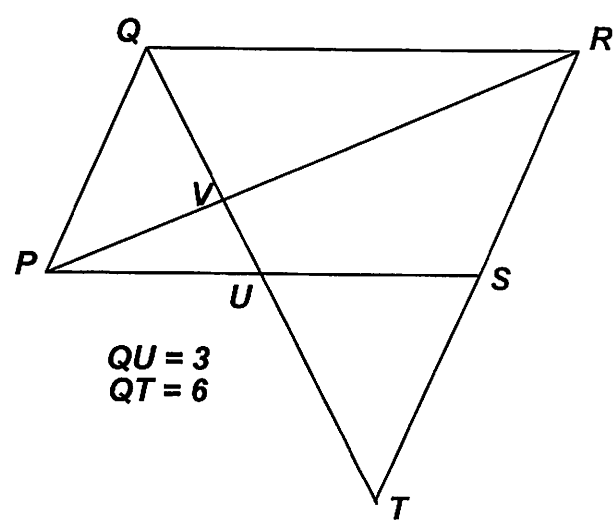


2. $m\angle DEC = 133° \Rightarrow$ major arc $\overset{\frown}{DC} = 266°$
   $\Rightarrow$ minor arc $\overset{\frown}{DC} = 94°$
   $m\angle BFA = 115° \Rightarrow$ major arc $\overset{\frown}{AB} = 230°$
   $\Rightarrow$ minor arc $\overset{\frown}{AB} = 130°$
   $\Rightarrow y + z = (360 - 94 - 130) = 136$
   As an angle formed by two chords,
   $x = m\angle BMD = \dfrac{y+z}{2} = 68°$
   As an angle formed by 2 secants,
   $m\angle P = 42 = \dfrac{1}{2}(y - z) \Rightarrow y - z = 84$
   Solving simultaneously, $\begin{cases} y + z = 136 \\ y - z = 84 \end{cases} \Rightarrow$
   $(x, y, z) = \underline{(\mathbf{68, 110, 26})}$.



3. By AA, $\Delta PQU \sim \Delta STU$ and $QU = TU \Rightarrow PQ = ST$
   Since $PQRS$ is a parallelogram, $PQ = RS$ and
   it follows that S must be a midpoint.
   By AA, $\Delta QVP \sim \Delta TVR$.
   Let $QV = x$ and $TV = 6 - x$. Then:
   $\dfrac{QV}{TV} = \dfrac{x}{6-x} = \dfrac{QP}{PR} = \dfrac{1}{2} \Rightarrow 2x = 6 - x \Rightarrow x = \underline{\mathbf{2}}$.

   Alternately, $\Delta QRV \sim \Delta UPV$ with $\dfrac{QR}{UV} = 2$
   $QV = x \Rightarrow \dfrac{x}{3-x} = 2$ and we have the same result.



Detailed Solutions for GBML Meet 5 – February 2015

ROUND 4

1. $\frac{x-1}{3x}-\frac{x}{x-1}+\frac{1}{x^2-x}<0 \Leftrightarrow$

$$\frac{(x-1)^2-x(3x)+3}{3x(x-1)}<0 \Rightarrow \frac{-2x^2-2x+4}{3x(x-1)}<0 \Rightarrow \frac{-2(x+2)\cancel{(x-1)}}{3x\cancel{(x-1)}}<0$$

Cancellation is possible, provided $x \neq 1$.

Multiplying through by $-\frac{3}{2}$, we have $\frac{(x+2)}{x}>0$.

The critical points are 0 and –2, and the quotient is positive to the left and to the right.

The solution set is $\{x<-2 \text{ or } x<0 \text{ and } x \neq 1\} \Rightarrow (A,B,C)=\underline{(\mathbf{-2,0,1})}$.

2. $\frac{a}{1-r}=6 \Rightarrow a=6(1-r)$

$a+ar=a(1+r)=\frac{9}{2}$

Substituting for a, $6(1-r)(1+r)=6(1-r^2)=\frac{9}{2} \Rightarrow 1-r^2=\frac{3}{4} \Rightarrow r=\pm\frac{1}{2}$

Therefore, $a=6\left(1\pm\frac{1}{2}\right) \Rightarrow \underline{\mathbf{3, 9}}$.

3. Let r and s be the roots of the quadratic $3x^2=J-Kx \Leftrightarrow 3x^2+Kx-J=0 \Leftrightarrow x^2+\frac{K}{3}x-\frac{J}{3}=0$

$\Rightarrow$ the sum of roots = $r+s=-\frac{K}{3}$ and the product of the roots = $rs=-\frac{J}{3}$

The first condition specifies that $r+s+1=-rs \Leftrightarrow \boxed{-\frac{K}{3}+1=\frac{J}{3}}$ (1)

The second condition specifies that $\frac{1}{r}\cdot\frac{1}{s}=\frac{5}{2}+\left(\frac{1}{r}+\frac{1}{s}\right) \Leftrightarrow \frac{1}{rs}=\frac{5}{2}+\left(\frac{r+s}{rs}\right) \Leftrightarrow \boxed{-\frac{3}{J}=\frac{5}{2}+\frac{K}{J}}$ (2)

(1) $\Rightarrow K=3-J$

(2) $\Rightarrow -6=5J+2K=5J+2(3-J) \Rightarrow 3J=-12 \Rightarrow J=-4$

Thus, $(K,J)=\underline{(\mathbf{7,-4})}$.

Detailed Solutions for GBML Meet 5 – February 2015

ROUND 5

1. $\sum_{k=1}^{k=19}(-1)^k(k)(i^{2k}) = -1\cdot i^2 + 2i^4 - 3i^6 + 4i^8 - \ldots - 19i^{38}$

   $1+2+3+4+\ldots+19 = \dfrac{19\cdot 20}{2} = \underline{\mathbf{190}}$

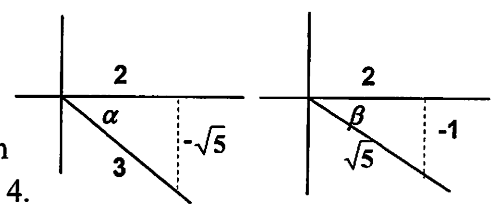
2. $\log\left(\cos\dfrac{5\pi}{3}\right)+\log\left(\cot\dfrac{7\pi}{6}\right)=\log(\tan\theta)+\log(\cos\theta)$

   $\Leftrightarrow \log\left(\cos\dfrac{5\pi}{3}\cdot\cot\dfrac{7\pi}{6}\right)=\log(\tan\theta\cdot\cos\theta)$

   $\Leftrightarrow \log\left(\dfrac{1}{2}\cdot\sqrt{3}\right)=\log(\sin\theta)$

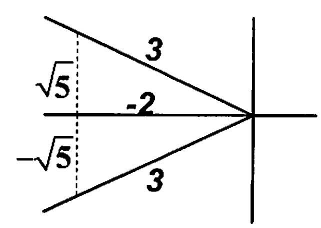
   $\Rightarrow \sin\theta=\dfrac{\sqrt{3}}{2}\Rightarrow\theta=\underline{\dfrac{\pi}{3},\dfrac{2\pi}{3}}$

3. Let $\alpha = Tan^{-1}\left(-\dfrac{\sqrt{5}}{2}\right)$ and $\beta = Sin^{-1}\left(-\dfrac{1}{\sqrt{5}}\right)$.



   Both α and β must be quadrant 4 values, since both principle values are only defined for quadrants 1 and 4.

   Clearly, $\sin\alpha=-\dfrac{\sqrt{5}}{3}$ and $\cos\beta=\dfrac{2}{\sqrt{5}}$, producing $\cos\theta=-\dfrac{2}{3}$.



   Since $\cos\theta<0,\theta$ must be in either quadrant 2 or quadrant 3.

   For $\theta\in Q2$, $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\dfrac{\sqrt{5}}{3}\cdot-\dfrac{2}{3}=-\dfrac{4\sqrt{5}}{9}$.

   For $\theta\in Q3$, $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot-\dfrac{\sqrt{5}}{3}\cdot-\dfrac{2}{3}=+\dfrac{4\sqrt{5}}{9}$.

   Thus, $\sin 2\theta = \underline{\pm\dfrac{4\sqrt{5}}{9}}$.

Detailed Solutions for GBML Meet 5 – February 2015

TEAM ROUND

1. $24x\left(\dfrac{2}{3x}+\dfrac{1}{8}=\dfrac{3x}{2}+8\right) \Rightarrow 16 + 3x = 36x^2 + 192x$

   $\Rightarrow 36x^2 + 189x - 16 = 0$

   Since the coefficient of the middle term is odd, the coefficients of the binomial factors must be aligned so that the outer and inner products are even and odd (or vice versa).
   This leaves only one possible factorization of 16, namely 1(16) and we have with minimal trial and error, $(12x - 1)(3x + 16) = 0$ and - (Both check!)

   Alternate Solution #1: (Substitution $a = 2/(3x)$)

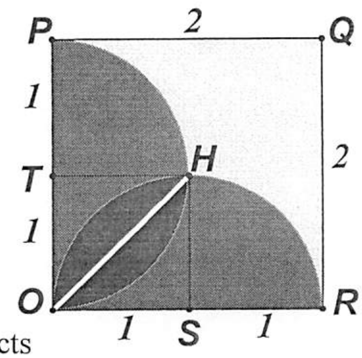
   $a+\dfrac{1}{8}=\dfrac{1}{a}+8 \Rightarrow 8a^2-63a-8=(8a+1)(a-8)=0 \Rightarrow a=-\dfrac{1}{8}, 8 \Leftrightarrow x=\underline{\mathbf{\dfrac{1}{12}, -\dfrac{16}{3}}}$

   Alternate Solution #2: (Combining terms on each side)

   $\dfrac{16+3x}{24x}=\dfrac{3x+16}{2}$. Since Equality is only possible if numerators are both zero or denominators are equal, we get the above answers.

2. *TOSH* is a square. The area of ΔHOT is $\dfrac{1}{2}$.

   The area of sector *THO* is $\dfrac{1}{4}\pi\cdot 1^2=\dfrac{\pi}{4}$. Thus, the area of the segment (the region between the chord $\overline{HO}$ and circle *T*) is $\dfrac{\pi}{4}-\dfrac{1}{2}$; so, the area common to the two circles is $\dfrac{\pi}{2}-1$.



   Subtracting the area of a full circle, from the area of the square, subtracts the overlap twice, so it must be added back in. The region *PQRH* is $4-\pi+\left(\dfrac{\pi}{2}-1\right)=3-\dfrac{\pi}{2}$

   Thus, the required ratio is $\dfrac{3-\dfrac{\pi}{2}}{\dfrac{\pi}{2}-1}=\dfrac{6-\pi}{\pi-2} \Rightarrow (A,B)=\underline{\mathbf{(6,2)}}$.

Detailed Solutions for GBML Meet 5 – February 2015

TEAM ROUND – continued

3. SAN – all different ANTONIO – 5 different (2 Os 2 Ns A,T,I – 1 each)
   Case 1: *S* picked from SAN and

   paired with 2 distinct letters from ANTONIO in $\binom{5}{2}=10$ ways $\Rightarrow$ 60 arrangements

   paired with 2 identical letters from ANTONIO 2 ways $\Rightarrow$ 6 arrangements Total: 66

   Case 2: A picked from SAN (only 1 letter from ANTONIO is a match) and

   paired with 2 distinct letters from ANTONIO (but not an A) $\binom{4}{2}=6$ ways $\Rightarrow$ 36 arrangements

   (ANT / ANO / ANI / ATO / ATI / AOI)
   paired with 2 distinct letters from ANTONIO (including an A) 4 ways $\Rightarrow$ 12 arrangements
   (AAN / AAT / AAO / AAI)
   paired with 2 identical letters from ANTONIO 2 ways $\Rightarrow$ 6 arrangements
   Total: 54
   (AOO / ANN)
   Case 3: N picked from SAN (2 letters from ANTONIO are a match) and
   paired with 2 Ns 1 way $\Rightarrow$ 1 arrangement (NNN)
   paired with 2 Os 1 way $\Rightarrow$ 3 arrangements (NOO)

   paired with 2 distinct letters (but no N) $\binom{4}{2}=6$ ways

   overlaps with case 2? ~~NAT~~ / ~~NAO~~ / ~~NAI~~ / NTO / NTI / NOI $\Rightarrow 3\cdot 6=18$ arrangements

   paired with 2 distinct letters (including 1 N) 4 ways
   overlaps with case 2? ~~NNA~~ / NNT / NNO / NNI $\Rightarrow 3\cdot 3=9$ arrangements
   Total: 31

   Grand total: <u>**151**</u>