GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2016

ROUND 1 - Arithmetic: Open

1. ____________________

2. ____________________

3. ____________________

1. The number of digits in the evaluation of $W = 2^{n+5} \cdot 3^2 \cdot 5^{n+1}$ in base 10 is k. Express k in terms of n.

2. When the base 10 integer N is written in base 7, it consists of 3 digits. When written in base 9, the same three digits appear, but in reverse order. Determine the tens digit of N (in base 10).

3. Let k denote the number of terminal zeros in $50!_{(\text{base } 72)}$. Compute k.

Detailed Solutions for GBML Meet 5 - FEBRUARY 2016

ROUND 1

1. $W = 2^{n+5} \cdot 3^2 \cdot 5^{n+1} = 2^{n+1} \cdot 5^{n+1} \cdot 2^4 \cdot 3^2 = 10^{n+1} \cdot 144$

   The number 10^{n+1} is 1 followed by $(n+1)$ terminal zeros.

   Multiplying by 144 results in 144 followed by $(n+1)$ terminal zeros.

   Thus, the expansion will have $\underline{n+4}$ digits.

2. Let $N_{(10)} = ABC_{(7)} = 49A + 7B + C = CBA_{(9)} = 81C + 9B + A$

   $\Leftrightarrow 48A - 2B - 80C = 0 \Leftrightarrow B = 8(3A - 5C)$

   Since $0 \le B \le 7$, $3A - 5C = 0 \Rightarrow (A, C) = (5, 3)$

   Thus, $N_{(10)} = 503_{(7)} = 49(5) + 7(0) + 3 = 248$ and the tens digit of N is $\underline{4}$.

   Check: $CBA_{(9)} = 3(9)^2 + 0(9) + 5 = 248$

3. In base 10, the number $k(10^n)$, where k is not divisible by 10, has n terminal zeros.

   In base 72, the number $k(72^n)$, where k is not divisible by 72, has n terminal zeros.

   Since $72 = 2^3 \cdot 3^2$, we must extract all the 2s and 3s from 50!, i.e. write 50! as $2^a \cdot 3^b \cdot N$, where a and b have maximum values and N is the product of the remaining primes. All even integers from 2 to 50 contribute a factor of 2, but some contribute more than 1. $(4 = 2^2, 8 = 2^3, 16 = 2^4, \ldots)$, so we evaluate the expression

   $P_2 = \left\lfloor \frac{50}{2} \right\rfloor + \left\lfloor \frac{50}{4} \right\rfloor + \left\lfloor \frac{50}{8} \right\rfloor + \left\lfloor \frac{50}{16} \right\rfloor + \left\lfloor \frac{50}{32} \right\rfloor + \left\lfloor \frac{50}{64} \right\rfloor + \ldots$, where $\lfloor\ \rfloor$ denotes greatest integer.

   $P_2 = 25 + 12 + 6 + 3 + 1 + 0 = 47$

   Similarly, $P_3 = \left\lfloor \frac{50}{3} \right\rfloor + \left\lfloor \frac{50}{9} \right\rfloor + \left\lfloor \frac{50}{27} \right\rfloor + \left\lfloor \frac{50}{81} \right\rfloor + \ldots = 16 + 5 + 1 + 0 = 22$

   Thus, $50! = 2^{47} 3^{22} N = (2^3 3^2)^{11} (2^{14} 3^0 N) = (72)^{11} k$, where k is not divisible by 72 and we have established that 50! has $\underline{11}$ terminal zeros in base 24.

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 2 - Algebra 1: Open

1. ______________________________

2. ______________________________

3. ______________________________

1. Compute the <u>sum</u> of all <u>distinct</u> <u>positive</u> values of k that make $12x^2 + kx + 2 = 0$ factorable over the integers.

2. Compute $\left\{ x \mid \frac{5x}{2+5x} - \frac{3}{1-x} + \frac{13x+8}{2+3x-5x^2} = 0,\ x \in \text{reals} \right\}$

3. A 10-gallon container had 8 gallons of a 72% alcohol solution. x gallons of a 48% alcohol solution was added. Pure alcohol was then added to fill the container. The final mixture was 71.1% alcohol. How many gallons of the 48% mixture were added?

Detailed Solutions for GBML Meet 5 - FEBRUARY 2016

ROUND 2

1. The possible factorizations are:
$$\begin{aligned}(12x+2)(x+1)=0 &\Rightarrow k=14\\(12x+1)(x+2)=0 &\Rightarrow k=25\\(6x+2)(2x+1)=0 &\Rightarrow k=10\\(6x+1)(2x+2)=0 &\Rightarrow k=14\\(4x+1)(3x+2)=0 &\Rightarrow k=11\\(4x+2)(3x+1)=0 &\Rightarrow k=10\end{aligned}$$

Notice that if k is even, a common factor of 2 could have been factored from each coefficient in the trinomial; hence, the duplication.
Thus, the sum is **<u>60</u>**.

2. $\frac{5x}{2+5x}-\frac{3}{1-x}+\frac{13x+8}{2+3x-5x^2}=0 \Leftrightarrow \frac{5x(1-x)-3(2+5x)+13x+8}{(2+5x)(1-x)}=0 \Leftrightarrow \frac{2+3x-5x^2}{(2+5x)(1-x)}=0$

$\Leftrightarrow \frac{(1-x)(2+5x)}{(2+5x)(1-x)}=0$

If $x=1$ or $x=-\frac{2}{5}$, the quotient is undefined; otherwise, the quotient equals 1. In either case, there is no value of x which satisfies the equation and the solution set is empty.

Acceptable answers: $\{\ \}$, null set, empty set, ϕ, There are no solutions.

Unacceptable: $\{\phi\}$

3. $0.72(8)+0.48x+1.00(10-8-x)=0.711(10)$

$\Leftrightarrow 576+48x+100(2-x)=711$

$\Leftrightarrow 65=52x$

$\Rightarrow x=\underline{\mathbf{\frac{5}{4}}}=\underline{\mathbf{1.25}}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2016

ROUND 3 - Geometry: Open

1. ____________________ °

2. __________ : __________

3. __________ : __________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

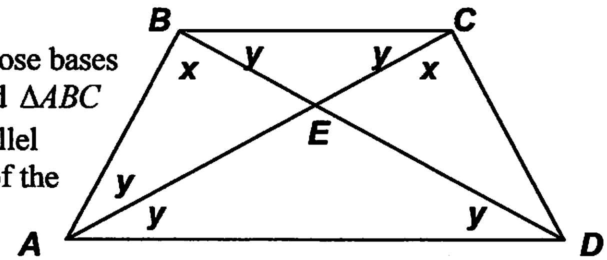
1. In quadrilateral *ABCD*, $AB = BC = CD$ and $AD = AC = BD$.
   Compute the degree-measure of $\angle ABC$.

2. In right triangle *ABC*, $\overline{CD}$ is the altitude to the hypotenuse, *P* is the midpoint of $\overline{AC}$, and *Q* is the midpoint of $\overline{BC}$. If $CA = 4$, $CB = 6$, and $\overline{CD}$ intersects $\overline{PQ}$ at point *R*, compute the ratio of the area of trapezoid *PRDA* to the area of trapezoid *QRDB*.

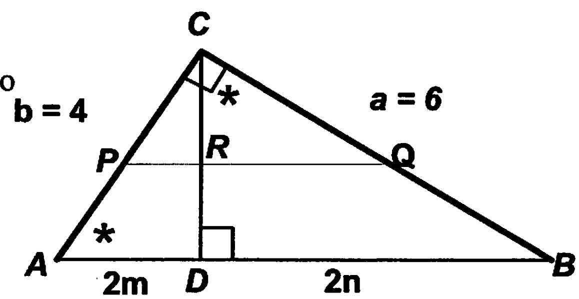
3. In right ΔPQR, *Q* is the vertex of the right angle, and the lengths of all the sides are integers. The ratio of the lengths of the legs is $t:(3t-1)$. Circle *O* in inscribed in ΔPQR. Compute the ratio of the length of the radius of circle *O* to the length of the side $\overline{PR}$.

Detailed Solutions for GBML Meet 5 - FEBRUARY 2016

ROUND 3



1. $ABCD$ must be an isosceles trapezoid, one of whose bases has the same length as its legs. Since ΔBEC and ΔABC are isosceles and alternate interior angles of parallel lines are congruent, let x and y be the measures of the indicated angles in the diagram. Since ΔBAD is isosceles, $x = 2y$ and $x + 3y = 180°$.
   Substituting for x, $5y = 180 \Rightarrow y = 36$, $x = 72$, $m(\measuredangle ABC) = \underline{\mathbf{108}}°$.

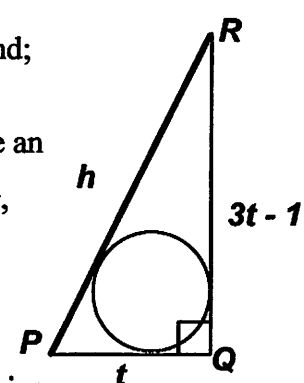


2. $\overline{PQ} \parallel \overline{AB} \Rightarrow \Delta PCR \sim \Delta ACD$, $\Delta QCR \sim \Delta BCD$
   Since P and Q are midpoints, in both cases, the ratio of the corresponding sides is 1 : 2.
   Therefore, $PR = m$ and $QR = n$. Let $CD = h$.

   The required ratio is $\dfrac{\frac{1}{2}\cdot\frac{h}{2}(m+2m)}{\frac{1}{2}\cdot\frac{h}{2}(n+2n)} = \dfrac{m}{n}$

   Since $m\angle A = m\angle DCB$, $\Delta DAC \sim \Delta DCB$, $\dfrac{DA}{DC} = \dfrac{DC}{DB} = \dfrac{AC}{CB} = \dfrac{2}{3} \Rightarrow \dfrac{2m}{h} = \dfrac{2}{3}$ and $\dfrac{h}{2n} = \dfrac{2}{3}$.

   Cross multiplying, $m = \dfrac{1}{3}h$, $n = \dfrac{3}{4}h \Rightarrow m : n = \underline{\mathbf{4:9}}$.



3. If you know your Pythagorean Triples well, 12-35-37 comes quickly to mind; otherwise, the discovery would be more time-consuming, as follows:
   Since all sides of ΔPQR are integers, then $h = PR = \sqrt{t^2 + (3t-1)^2}$ must be an integer. Thus, the radicand must be a perfect square, i.e., for some integer n,
   $t^2 + (3t-1)^2 = n^2$. $t = 1,2,3,\ldots \Rightarrow n^2 = 5, 29, 73, 137, 221, \ldots$
   The gaps are 24, 44, 64, 84,…
   The gaps are growing by 20 each time.
   Thus, we can generate the n^2-sequence by adding 104, 124, 144, … . , stopping when we find a perfect square. 325, 449, 593, 757, 941, 1145, **1369** ($=37^2$)
   Invoking $r_{ic} = \dfrac{A(\Delta)}{s}$, where s denotes the semi-perimeter, the radius of the inscribed circle O

   is $\dfrac{\frac{1}{2}\cdot 12\cdot 35}{\frac{12+35+37}{2}} = \dfrac{12\cdot 35}{84} = 5$. Thus, the required ratio is **<u>5 : 37</u>**.

   How do we know this is a unique solution?

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 4 – Algebra 2: Open

1. ______________________________

2. ______________________________

3. ______________________________

1. Given: $F(n+1)=\dfrac{2F(n)+1}{2}$ for $n=1,2,3,\ldots$ and $F(1)=2$.
   Compute $F(100)$.

2. Two roots of the equation $x^3+px+q=0$ are $a\pm bi$, where $b\neq 0$, $i=\sqrt{-1}$, and p, q, a, and b are real numbers. Express p in terms of a and b.

3. The solution(s) over the reals of $\log_{x^2}(9x)+\log_{x^3}(8x)=\log_6 x$ may be expressed in the form $x=a^n$, where a denotes a minimum positive integer constant and n is a rational number. Compute all possible ordered pairs (a,n).

Detailed Solutions for GBML Meet 5 - FEBRUARY 2016

ROUND 4

1. Evaluating the first few terms in the sequence defined recursively by $F(n+1)=\frac{2F(n)+1}{2}$ and the initial condition $F(1)=2$, we have $F(2)=\frac{5}{2}$, $F(3)=3$, $F(4)=\frac{7}{2}$, $F(5)=4,\ldots$. We note the pattern and realize there is a much simpler explicit formula for $F(n)$, namely $F(n)=\frac{n+3}{2}$.
Thus, $F(100)=$ $\underline{\mathbf{\frac{103}{2}}\text{ or }\mathbf{51.5}}$.

2. The coefficient of x^2 is the opposite of the sum of the roots. Let the third root be k.
Then $-(k+a+bi+a-bi)=0 \Rightarrow k=-2a$
The coefficient of x is the sum of the products of the roots taken two at a time.
$p=-2a(a+bi)-2a(a-bi)+(a+bi)(a-bi)=-4a^2+a^2+b^2 = \underline{\mathbf{b^2-3a^2}}$.

3. Using the base conversion rules, $\log_a b=\frac{1}{\log_b a}$ and $\log_{a^n}(b)=\frac{1}{n}\log_a b$, we have

$$\log_{x^2}(9x)+\log_{x^3}(8x)=\log_6 x \Leftrightarrow \frac{1}{2}\log_x(9x)+\frac{1}{3}\log_x(8x)=\frac{1}{\log_x 6}$$

$$\Leftrightarrow \frac{1}{2}\cdot 2\log_x 3+\frac{1}{3}\cdot 3\log_x 2+\left(\frac{1}{2}+\frac{1}{3}\right)=\frac{1}{\log_x 6}$$

$$\Leftrightarrow \log_x 3+\log_x 2+\frac{5}{6}=\frac{1}{\log_x 6} \Leftrightarrow \log_x 6+\frac{5}{6}=\frac{1}{\log_x 6}$$

$$\Leftrightarrow 6(\log_x 6)^2+5(\log_x 6)-6=0$$

$$\Leftrightarrow (3\log_x 6-2)(2\log_x 6+3)=0 \Rightarrow \log_x 6=\frac{2}{3},-\frac{3}{2} \Rightarrow \log_6 x=\frac{3}{2},-\frac{2}{3} \Rightarrow (a,n)=\underline{\left(\mathbf{6,\frac{3}{2}}\right),\left(\mathbf{6,-\frac{2}{3}}\right)}.$$

Note: $6^{\frac{3}{2}}=36^{\frac{3}{4}}=\ldots$ and $6^{-\frac{2}{3}}=36^{-\frac{1}{3}}=\ldots$, but since we require that a be a minimum positive integer, the stated solutions are unique.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2016

ROUND 5 - Pre-Calculus: Open

1. ______________________________

2. _____________ : _____________

3. ______________________________

1. The polynomial $P(x)=x+x^9+x^{25}+x^{125}$ is divided by the polynomial $D(x)=x^3-x$. Compute the remainder polynomial $R(x)$.

2. In the expansion of $(Ax+B)^{11}$, where $A>0$ and $B>0$, the ratio of the coefficient of the 8^{th} term to the coefficient of the 10^{th} term is 32 : 27. Compute the ratio $A:B$.

3. ΔABC is a 3-4-5 right triangle. Compute the length of the trisector of the right angle that is nearer the shorter leg of the right triangle.

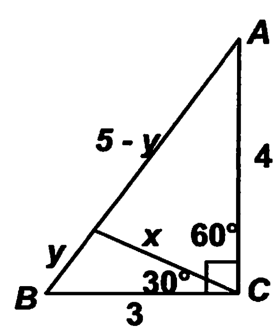
Detailed Solutions for GBML Meet 5 - FEBRUARY 2016

ROUND 5

1. Suppose the division produces a quotient $Q(x)$ and a remainder $R(x)$.
Since the remainder polynomial must be of a smaller degree than the divisor polynomial, we let $R(x) = Ax^2 + Bx + C$
Note that $D(x) = 0$, for $x = 0, \pm 1$
Since $P(x) = D(x)\cdot Q(x) + R(x)$, we do not need to know the specific formula for $Q(x)$, since, when evaluating for $x = 0, \pm 1$, $Q(x)$ will be multiplied by 0.
Thus, $P(0) = R(0)$, $P(1) = R(1)$ and $P(-1) = R(-1)$.
Substituting, $\begin{cases} C = 0 \\ A + B = 4 \\ A - B = -4 \end{cases} \Rightarrow (A,B) = (0,4)$ and $R(x) = \underline{\mathbf{4x}}$.

2. The 8[th] and 10[th] terms in the expansion are $\binom{11}{7}A^4B^7$ and $\binom{11}{9}B^2A^9$ respectively.
Note that $\binom{11}{9} = \binom{11}{2} = \frac{11\cdot 10}{1\cdot 2}$ and $\binom{11}{7} = \binom{11}{4} = \frac{11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdot 3\cdot 4}$
Thus, $\frac{\binom{11}{7}A^4B^7}{\binom{11}{9}A^2B^9} = \frac{\cancel{11\cdot 10}\cdot 9\cdot 8}{\cancel{1\cdot 2}\cdot 3\cdot 4}\cdot\frac{\cancel{1\cdot 2}}{\cancel{11\cdot 10}}\cdot\frac{A^2}{B^2} = \frac{6A^2}{B^2} = \frac{32}{27} \Rightarrow \frac{A^2}{B^2} = \frac{16}{81} \Rightarrow \frac{A}{B} = \underline{\mathbf{\frac{4}{9}}}$

3. In ΔABC, $\frac{\sin 30°}{y} = \frac{\sin C}{x}$ and $\frac{\sin 60°}{5-y} = \frac{\sin A}{x}$
$\Leftrightarrow \frac{1}{2y} = \frac{4}{5x}$ and $\frac{\sqrt{3}}{2(5-y)} = \frac{3}{5x}$
From the first condition, $y = \frac{5}{8}x$.
Substituting in the second condition and cross multiplying,



$5\sqrt{3}x = 6\left(5 - \frac{5}{8}x\right) \Leftrightarrow \sqrt{3}x = 6\left(1 - \frac{1}{8}x\right)$
$\Leftrightarrow x\left(8\sqrt{3} + 6\right) = 48 \Leftrightarrow x = \frac{24}{4\sqrt{3}+3}\cdot\frac{4\sqrt{3}-3}{4\sqrt{3}-3} \Leftrightarrow \frac{24}{48-9}\left(4\sqrt{3}-3\right) = \underline{\mathbf{\frac{8}{13}\left(4\sqrt{3}-3\right)}}$ or $\underline{\mathbf{\frac{32\sqrt{3}-24}{13}}}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2016

TEAM ROUND

3 pts.	1.	______________________
3 pts.	2.	______________________
4 pts.	3.	______________________

1. A stack of chocolate cannonballs forms a "pyramid" (1 cannonball on top, 4 in the second layer, 9 in the third, etc.). Each cannonball is spherical with a radius of 6mm. The pyramid is resting on a cookie sheet. If the maximum distance from a point on the surface of the cannonball in the top layer to the cookie sheet must be less than 60mm, how many layers of chocolate cannonballs are there?

2. Solve for x over the reals. $(x+3)^3(x-1)^3+(x-2)^3(x-1)^3=(2x+1)^3(x-1)^3$

3. A number N is randomly selected from the set of five-digit natural numbers (in base 10) whose digits sum to 43. Compute the probability that the number selected will be divisible by 11.

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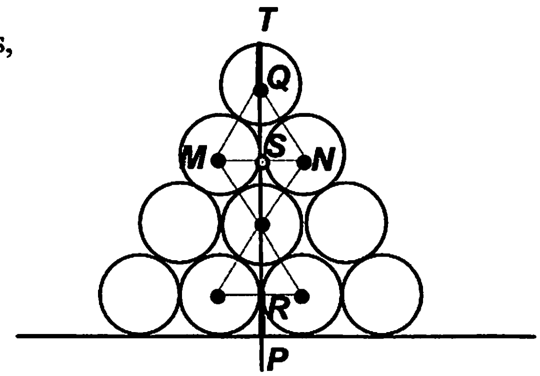
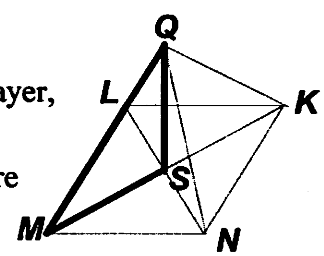
TEAM ROUND

1. Let T be the point furthest from the cookie sheet.
   The diagram at the right is a cross-section of the pyramid formed by a plane perpendicular to the cookie sheet and containing point T.
   Note that each of the triangles formed by the dashed lines is an equilateral triangle with side $2r$.
   The distance TP is comprised of 2 radii and 3 altitudes. Thus,
   $TP = 2r + 3h = 2r + 3\left(r\sqrt{3}\right)$



   For n layers, $TP = 2r + (n-1)h = 2r + (n-1)\left(r\sqrt{3}\right) < 60$.
   $r = 6 \Rightarrow (n-1)\sqrt{3} < 8$
   $n = 5 \Rightarrow 4\sqrt{3} < 8$, since $\left(4\sqrt{3}\right)^2 = 48 < 60$, but
   $n = 6 \Rightarrow 5\sqrt{3} > 8$, since $\left(5\sqrt{3}\right)^2 = 75 > 60$. Thus, $n = 5$.
   Pretty convincing! HOWEVER, this *two dimensional* analysis of a *three dimensional* problem misses the mark.
   In the analysis above, $\overline{QS}$ was treated as the altitude of an equilateral triangle, when, in fact, it is the altitude of a pyramid with a square base. [Q is the vertex of the pyramid and M and N are two of the 4 vertices of the square base, i.e. the centers of the spheres in the second layer of cannonballs.]
   Let Q be the center of the sphere in the top layer.
   Let K, L, M and N be the centers of the spheres at the corners of the n^{th} layer, which contains n^2 spheres. Since the spheres have 6" radii,
   $QM = MN = 12(n-1)$. Since S is the midpoint of the diagonals of square $KLMN$, $MS = 6(n-1)\sqrt{2}$. Applying the Pythagorean Theorem,



   $QS = \sqrt{12^2(n-1)^2 - 6^2(n-1)^2 2} = \sqrt{6^2(n-1)^2(4-2)} = 6(n-1)\sqrt{2}$
   Thus, $TP = 12 + 6(n-1)\sqrt{2} < 60 \Leftrightarrow (n-1)\sqrt{2} < 8$.
   Squaring both sides, $n = 5 \Rightarrow 32 < 64$, $n = 6 \Rightarrow 50 < 64$, but $n = 7 \Rightarrow 72 \not< 64$.
   Thus, the maximum number of layers is **<u>6</u>**.

TEAM ROUND – continued

2. Clearly, $x=\underline{1}$ is a solution (with multiplicity 3). Dividing through by $(x-1)^3$, the equation simplifies to $(x+3)^3+(x-2)^3=(2x+1)^3$. Notice that $(x-2)+(x+3)=2x+1$.
   Substituting $u=x-2$ and $v=x+3$, the original equation is equivalent to $u^3+v^3=(u+v)^3$.
   Expanding the right hand side and cancelling, we have $3u^2v+3uv^2=3uv(u+v)=0$.
   Thus, $x-2=0$, $x+3=0$, and $2x+1=0$ and we have the additional solutions of $x=\underline{\mathbf{2, -3, -\frac{1}{2}}}$.

   Brute Force solution (after extracting the common factor of $(x-1)^3$ and a solution of $x=1$)
   $(x+3)^3+(x-2)^3=(x^3+9x^2+27x+27)+(x^3-6x^2+12x-8)=2x^3+3x^2+39x+19$.
   $(2x+1)^3=8x^3+12x^2+6x+1$
   Equating, transposing terms and factoring, we have $3(2x^3+3x^2-11x-6)=0$
   The only possible rational roots are $\pm\left(1,2,3,6,\frac{1}{2},\frac{3}{2}\right)$
   By synthetic or direct substitution, $x=2$ is a solution and
   $2x^3+3x^2-11x-6=(x-2)(2x^2+7x+3)$.
   Factoring the quadratic, we have $(x-2)(x+3)(2x+1)=0$ and $x=\underline{\mathbf{1, 2, -3, -\frac{1}{2}}}$.

TEAM ROUND - continued

3. The digits of the five-digit numbers consists of four 9s and one 7 or three 9s and two 8s.
For a five-digit number *abcde* to be divisible by 11, the difference of the sums of alternate digits must be a multiple of 11, i.e. $(a+c+e)-(b+d)=11k$

Case 1: 9,9,9,9,7
The sum of any 3 digits must be either 27 or 25, with the corresponding 2-digit sums of 16 or 18.
Only $27-16$ produces a multiple of 11 and the possible five-digit numbers are 99979 and 97999

Case 2: 9,9,9,8,8
The possible digit differences are $27-16$, $26-17$, and $25-18$
Only the first difference is a multiple of 11 and this corresponds to 98989
Total divisible by 11: 3

There are $\frac{5!}{4!\cdot 1!}=5$ permutations of 99998.

There are $\frac{5!}{2!\cdot 3!}=10$ permutations of 99988.

Thus, the required probability is $\frac{3}{5+10}=\frac{\mathbf{1}}{\underline{\mathbf{5}}}$.

GREATER BOSTON MATHEMATICS LEAGUE

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Answer Sheet

Round 1

1. $n+4$
2. 4
3. 11

Round 2

1. 60
2. ϕ or $\{\ \}$, null set
   Unacceptable: $\{\phi\}$
3. $\frac{5}{4}$ or 1.25 gal

Round 3

1. 108°
2. 4:9
3. 5:37

Round 4

1. $\frac{103}{2}$ or 51.5
2. b^2-3a^2
3. $\left(6,\frac{3}{2}\right),\left(6,-\frac{2}{3}\right)$

Round 5

1. $4x$
2. 4 : 9
3. $\frac{32\sqrt{3}-24}{13}$

Team Round

1. 6 (3 pts)
2. $1, 2, -3, -\frac{1}{2}$ (3 pts)
3. $\frac{1}{5}$ (4 pts)