GREATER BOSTON MATHEMATICS LEAGUE

MEET 5-FEBRUARY2018

ROUND 1-Arithmetic: Open

1. ______________________

2. ______________________

3. ______________________

1. Given: P, Q, and W are distinct positive integers for which $P + \cfrac{1}{Q + \cfrac{2}{W}} = \dfrac{97}{30}$.

   Compute the product PQW.

2. Compute the 78th natural number that is divisible by either 3 or 5, but not divisible by either 6 or 15.

3. 15! (factorial) written in bases 10, 12 and 18 ends in P, Q, and R zeros respectively. Compute the sum $P + Q + R$ (in base 10).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 - FEBRUARY 2018

ROUND 2-Algebra 1: Open

1. P: \$_______ Q: \$_______

2. ________________________

3. ________________________

1. Pete initially has 2/3 as much money as Quincy. Pete gives 1/4 of his money to Quincy.
   After this transfer, Quincy gives 1/3 of his money back to Pete.
   Pete now has \$4 more than Quincy.
   How much money (in dollars) did each boy have initially?

2. Simplify completely. Your expression must use a minimum number of minus signs.

$$\left(\frac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}}\right)\div\left(\frac{1}{y}-\frac{1}{x}\right)^2$$

3. The three-digit base 9 number ABC equals the three-digit base 7 number BCA. Compute all possible ordered triples (A,B,C) for which this is true.

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 3-Geometry: Open

1. ____________________

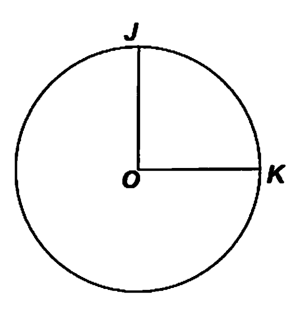
2. ____________________

3. (__________ , __________)

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. $ABCD$ is a trapezoid with $\overline{AB}$ as a base. Let $(AB, BC, CD, DA) = (40, 15, 15, 20)$. Perpendiculars from D and C intersect $\overline{AB}$ in points E and F respectively. Compute the absolute value of the difference of the areas of ΔDAE and ΔCFB.

2. An equilateral triangle ABC is inscribed in a 90° sector of circle O. Point A is the midpoint of arc $\overset{\frown}{JK}$. Compute the ratio of the area of ΔABC to the area of ΔOBC.



3. The sides of a triangle have lengths 24, 14, and 18. The <u>largest</u> angle in the triangle is bisected forming two triangles. The areas of these triangles are J and K, where $J<K$. Compute the ordered pair (J, K).

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 4 – Algebra 2: Open

1. (__________ , __________)

2. ______________________________

3. (________ , ________ , ________)

1. Compute the ordered pair (*a*, *b*) for which $2x^3 - x^2 + ax + b$ is divisible by $(x-2)(x+4)$.

2. How many <u>integers</u> *x* satisfy the following system of inequalities?

$$\begin{cases} |2-3x| < 13 \\ x^2 - x \geq 2 \end{cases}$$

3. The equation $27x^3 + 9x = 10 - 54x^2$ has three roots which form an arithmetic sequence r_1, r_2, r_3. Compute the ordered triple (r_1, r_2, r_3), where $r_1 < r_2 < r_3$.

GREATER BOSTON MATHEMATICS LEAGUE

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ROUND 5 – Pre-Calculus: Open

1. ______________________

2. ______________________

3. ______________________

1. Given: $\tan(2x) = \frac{24}{7}$, where $0° \le x < 360°$.
   Compute all possible values of $\sin^2 x - \cos^2 x$.

2. How many different four-letter arrangements are possible using the letters from the word SESAME?

3. Given: $t = \log_{12}\left(\sqrt{2}\right)$
   Determine a simplified expression, in terms of t, for $\log_6\left(\frac{9}{4} + \sqrt{0.5625}\right) - \log_6\left(\frac{9}{4} - \sqrt{0.5625}\right)$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 - FEBRUARY 2018

TEAM ROUND

3 pts.	1.	______________________
3 pts.	2.	(__________ , __________)
4 pts.	3.	______________________

1. Compute all values of x which satisfy

$$\frac{x}{2x-1}+\frac{x}{2x+1}+\frac{2x+1}{1-2x}=\frac{2x^2+5x}{1-4x^2}.$$

2. If the three coefficients of the quadratic equation $x^2+bx+c=0$ are increased by 1, the roots are also increased by 1. Compute the ordered pair (b, c).

3. The coordinates of the midpoints of the sides of $\Delta V_1V_2V_3$ are $P(0,2)$, $Q(4,-2)$, and $R(2,1)$ Let O be the origin of the coordinate plane. Compute the *maximum* possible area of ΔV_jOV_k, where j and k are integers, $1 \le j \le 3$, and $1 \le k \le 3$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5-FEBRUARY2018

Answer Sheet

<u>Round 1</u>

1. 84
2. 291
3. 11

<u>Round 2</u>

1. P: $24 Q: $36
2. $\dfrac{xy}{y-x}$
3. (1,2,0), (2,4,0), (3,6,0)

<u>Round 3</u>

1. 42
2. $\sqrt{3}:1$
3. $\left(\dfrac{49}{2}\sqrt{5}, \dfrac{63}{2}\sqrt{5}\right)$

<u>Round 4</u>

1. $(-26, 40)$
2. 6
3. $\left(-\dfrac{5}{3}, -\dfrac{2}{3}, \dfrac{1}{3}\right)$

<u>Round 5</u>

1. $\pm\dfrac{7}{25}$ (or ± 0.28)
2. 102
3. $\dfrac{2t}{1-2t}$

<u>Team Round</u>

1. -1 (3 pts)
2. (5, 9) (3 pts)
3. 12 (4 pts)

Detailed Solutions for GBML Meet 5 -FEBRUARY2018

ROUND 1

1. For positive integers Q and W, $0<\dfrac{1}{Q+\dfrac{2}{W}}\le 0.5$.[$(Q,W)=(1,2)\Rightarrow\dfrac{1}{2}$. For other ordered pairs of positive integers, the denominator is larger; hence, the value of the fraction will be smaller.]

   Since $\dfrac{97}{30}=3\dfrac{7}{30}$, we take $\left(P,\dfrac{1}{Q+\dfrac{2}{W}}\right)=\left(3,\dfrac{7}{30}\right)\Rightarrow Q+\dfrac{2}{W}=\dfrac{30}{7}=4\dfrac{2}{7}$ and

   we have $(Q,W)=(4,7)$. Thus, the product $PQW=\underline{\mathbf{84}}$.

2. Condition: Divisible by 3 or 5, but not by 6 or 15
   Since the LCM of 3, 5, 6 and 15 is 30, we determine how many integers from 1 to 30 inclusive satisfy the given conditions. There are 8 such numbers, namely,
   3, 5, 9, 10, 20, 21, 25 and 27. In subsequent groups of 30 integers (31…60, 61…90, etc), there will also be 8 numbers satisfying the given conditions.
   Thus, the first 9 groups of 30 contain the first 72 compliant numbers.
   We need 6 more. The first of these will be 3 more than the last number in the 9th group.
   Similarly, the 6th will be 21 more than the last number in the 9th group.
   Therefore the 78th compliant natural number is $9\cdot 30+21=\underline{\mathbf{291}}$.

3. $15!=1\cdot 2\cdot 3\cdot 4\cdot\ldots\cdot 15=2^{11}\cdot 3^{6}\cdot 5^{3}\cdot 7^{2}\cdot 11\cdot 13$
   In base 10, we need to express this as $10^{n}\cdot X$, where n is as large as possible. Clearly, the 5s the limiting factor. We need one 2 for every 5, so $n = 3 \Rightarrow$ 15! ends in $P = 3$ zeros.
   In base 12, a numeral ends in a 0, if it is a power of 12 (instead of a power of 10).
   Since $12=2^{2}\cdot 3$, we need twice as many 2s as 3s.
   We can use ten 2s, but only five 3s$\Rightarrow Q = 5$ zeros.
   In base 18 $\left(=2\cdot 3^{2}\right)$, we are interested in numbers which are powers of 18, so we need twice as many 3s as 2s. We can use all six 3s, but only three 2s $\Rightarrow R = 3$ zeros.
   Thus, $P+Q+R=\underline{\mathbf{11}}$.

Detailed Solutions for GBML Meet 5 - FEBRUARY 2018

ROUND 2

1.

Pete	$4x$	$4x - x = 3x$	$3x + (7/3)x = (16/3)x$
Quincy	$6x$	$6x + x = 7x$	$7x - (7/3)x = (14/3)x$

Thus, $\frac{16}{3}x = \frac{14}{3}x + 4 \Rightarrow \frac{2}{3}x = 4 \Rightarrow x = 6 \Rightarrow P: \$\underline{24} \;\; Q: \$\underline{36}$.

2. $$\left(\frac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}}\right) \div \left(\frac{1}{y} - \frac{1}{x}\right)^2 \Leftrightarrow \left(\frac{\frac{y^2 - x^2}{x^2y^2}}{\frac{y+x}{xy}}\right) \div \frac{(x-y)^2}{x^2y^2} \Leftrightarrow \frac{y-x}{xy} \cdot \frac{x^2y^2}{(x-y)^2}$$

$$\Leftrightarrow \frac{-(x-y)}{1} \cdot \frac{xy}{(x-y)^2} = \frac{-xy}{x-y} = \underline{\frac{xy}{y-x}}$$,provided $x \neq 0$, $y \neq 0$, and $y \neq \pm x$.

3. $$\begin{cases} ABC_9 = 81A + 9B + C \\ BCA_7 = 49B + 7C + A \end{cases} \Rightarrow 80A - 40B = 6C \Rightarrow C = \frac{20(2A - B)}{3}.$$

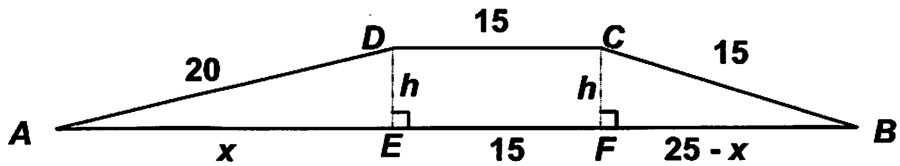
Since 3 is not a factor of 20, it must be a factor of $2A - B$.

$(A,B) = (1,2), (2,4), (3,6) \Rightarrow C = 0$

Since $0 \le A,B,C \le 6$ are the only allowable digits in both base 7 and 9, we have only 3 possible ordered triples, namely, $\underline{\mathbf{(1,2,0), (2,4,0), (3,6,0)}}$.

Detailed Solutions for GBML Meet 5 - FEBRUARY 2018

ROUND 3

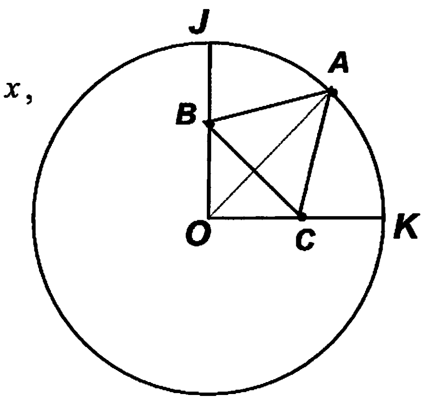


1. $\begin{cases} h^2 + x^2 = 400 \\ (25-x)^2 + h^2 = 225 \end{cases}$

$\Rightarrow 625 - 50x = -175$

$\Rightarrow x = 16, y = 12$

Thus, the required difference is $\frac{1}{2}\cdot 16\cdot 12 - \frac{1}{2}\cdot 9\cdot 12 = 6(16-9) = \underline{\mathbf{42}}$.



2. Assuming ΔBOC is an isosceles right triangle and $AB = BC = CA = x$, we have $OB = OC = \frac{x}{\sqrt{2}}$. Thus, the required areas are $\frac{x^2\sqrt{3}}{4}$ and $\frac{1}{2}\left(\frac{x}{\sqrt{2}}\right)^2 = \frac{x^2}{4}$ and the required ratio is $\underline{\boldsymbol{\sqrt{3}:1}}$.

Is it possible that $OB \neq OC$? Let the coordinates of A be (n,n).

Let's try $B(0,b)$ and $C(c,0)$, where $b \neq c$.

Since $AB^2 = AC^2$, we have $n^2 + (n-b)^2 = (n-c)^2 + n^2 \Rightarrow -2nb + b^2 = -2nc + c^2$

$\Rightarrow 2n(c-b) = c^2 - b^2 \Rightarrow 2n = c + b$. (Since $b \neq c$, we are *not* dividing by zero.)

Since $BC^2 = AC^2$, we have $b^2 + c^2 = (n-c)^2 + n^2 \Rightarrow b^2 = 2n^2 - 2nc = 2n(n-c)$

$= (c+b)\left(\frac{c+b}{2} - c\right) = \frac{(b+c)(b-c)}{2} = \frac{b^2 - c^2}{2} \Rightarrow 2b^2 = b^2 - c^2 \Rightarrow b^2 = -c^2$, which could only be true if $b = c = 0$. Since our supposition was $b \neq c$, we have a contradiction. Thus, it must have been true that $b = c$ and our assumption that ΔBOC was an isosceles right triangle is justified.

3. By Heron's formula, the area of this triangle is $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$.

$s = \frac{24+14+18}{2} = 28 \Rightarrow A = \sqrt{28\cdot 4\cdot 14\cdot 10} = \sqrt{2^6\cdot 7^2\cdot 5} = 56\sqrt{5}$

The largest angle is opposite the longest side. Assuming this side is divided into segments of length x and $24 - x$, we apply the angle bisector theorem,

$\frac{14}{x} = \frac{18}{24-x} \Rightarrow 9x = 168 - 7x \Rightarrow x = 10.5$. Thus, the bases of the two triangles are in a ratio of 10.5 : 13.5 = 21 : 27 = 7 : 9 and the two triangles must have areas of

$\left(\frac{7}{16}\cdot 56\sqrt{5}, \frac{9}{16}\cdot 56\sqrt{5}\right) = \underline{\left(\mathbf{\frac{49}{2}\sqrt{5}, \frac{63}{2}\sqrt{5}}\right)}$.

Detailed Solutions for GBML Meet 5 - FEBRUARY 2018

ROUND 4

1. Suppose $\dfrac{2x^3-x^2+ax+b}{(x-2)(x+4)}=Q(x)$ with no remainder.

   Then: $2x^3-x^2+ax+b=(x-2)(x+4)Q(x)$

   Letting $x=2$, we have $16-4+2a+b=0 \Leftrightarrow 2a+b=-12$.

   Letting $x=-4$, we have $-128-16-4a+b=0 \Leftrightarrow -4a+b=144$.

   Subtracting, $6a=-156 \Rightarrow a=-26 \Rightarrow (a,b)=\underline{\mathbf{(-26,40)}}$.

2. $\begin{cases} |2-3x|<13 \\ x^2-x\geq 2 \end{cases} \Leftrightarrow \begin{cases} -13<2-3x<13 \\ x^2-x-2=(x-2)(x+1)\geq 0 \end{cases}$

   The first inequality is equivalent to $-15<-3x<11 \Leftrightarrow 5>x>-\dfrac{11}{3}$.

   The second inequality is equivalent to $x\leq -1$ or $x\geq 2$.

   Taking the intersection, $-\dfrac{11}{3}<x\leq -1$ or $2\leq x<5$.

   Thus, the integer solutions are: -3, -2, -1, 2, 3, and 4, for a total of **6** solutions.

3. Given: $P(x)=27x^3+9x=10-54x^2 \Leftrightarrow 27x^3+54x^2+9x-10=0$

   Assuming the roots (which form an arithmetic sequence) are $r-a$, r and $r+a$

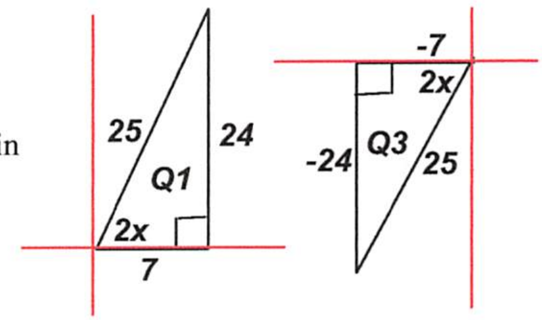
   Then the sum of the roots is $3r=-\dfrac{54}{27}=-2 \Rightarrow r=-\dfrac{2}{3}$

   By synthetic division, $P(x)=(3x+2)(9x^2+12x-5)$.

   Factoring the trinomial, $P(x)=(3x+2)(3x-1)(3x+5)=0 \Rightarrow (r_1,r_2,r_3)=\underline{\left(-\mathbf{\frac{5}{3}},-\mathbf{\frac{2}{3}},\mathbf{\frac{1}{3}}\right)}$.

Detailed Solutions for GBML Meet 5 - FEBRUARY 2018

ROUND 5



1. $\sin^2 x - \cos^2 x = -\cos(2x)$

   $\tan(2x) = \dfrac{24}{7} \Rightarrow$(7, 24, 25) and $2x$ denotes a value in Q1 or Q3. Thus, $-\cos(2x) = \underline{\pm\dfrac{\mathbf{7}}{\mathbf{25}}}$.

   (SOH-**<u>CAH</u>**-TOA)

2. Here are the possible foursomes.

   Two pairs of identical letters: $SSEE \Rightarrow \dfrac{4!}{2!\cdot 2!} = 6$

   Exactly one pair of identical letters: $SSEA$, $SSEM$, $SSAM$, $EESA$, $EESM$, $EEAM$

   Each $\dfrac{4!}{2!} = 12 \Rightarrow 6\cdot 12 = 72$

   All different letters: SEAM$\Rightarrow 4! = 24$.
   Thus, the total is 6 + 72+24 = **<u>102</u>**.

3. $t = \log_{12}\left(\sqrt{2}\right) = \dfrac{1}{2}\log_{12} 2 \Leftrightarrow \dfrac{1}{2t} = \log_2 12 = \log_2\left(2^2\cdot 3\right) = 2 + \log_2 3 \Rightarrow \log_2 3 = \dfrac{1-4t}{2t}$

   Since $\sqrt{0.5625} = \sqrt{\dfrac{9}{16}} = \dfrac{3}{4}$, we have

   $$\log_6\left(\frac{9}{4} + \sqrt{0.5625}\right) - \log_6\left(\frac{9}{4} - \sqrt{0.5625}\right) = \log_6\left(\frac{\frac{9}{4}+\frac{3}{4}}{\frac{9}{4}-\frac{3}{4}}\right) = \log_6 2$$

   Since $\log_6 2\cdot\log_2 6 = 1$, $\log_6 2 = \dfrac{1}{\log_2(2\cdot 3)} = \dfrac{1}{1+\log_2 3} = \dfrac{1}{1+\dfrac{1-4t}{2t}} = \underline{\dfrac{\mathbf{2t}}{\mathbf{1-2t}}}$.

Detailed Solutions for GBML Meet 5 - FEBRUARY 2018

TEAM ROUND

1. $\frac{x}{2x-1}+\frac{x}{2x+1}+\frac{2x+1}{1-2x}=\frac{2x^2+5x}{1-4x^2} \Rightarrow x \neq \pm\frac{1}{2}$

   Multiplying through by $(2x-1)(2x+1)$, we have

   $x(2x+1)+x(2x-1)-(2x+1)^2=-(2x^2+5x)$

   $\Leftrightarrow 2x^2+x+2x^2-x-4x^2-4x-1=-2x^2-5x \Leftrightarrow 2x^2+x-1=\cancel{(2x-1)}(x+1)=0 \Rightarrow x=\underline{-1}$.

2. Applying the root-coefficient relationship of quadratic equations, if the roots of $x^2+bx+c=0$ are r_1 and r_2, then $\begin{cases} r_1+r_2=-b \\ r_1r_2=c \end{cases}$.

   Now consider the roots r_3 and r_4 of the equation $2x^2+(b+1)x+(c+1)=0$.

   $r_3+r_4=(r_1+1)+(r_2+1)=-b+2=\frac{-(b+1)}{2} \Rightarrow -2b+4=-b-1 \Rightarrow b=5$

   $r_3 \cdot r_4=(r_1+1)(r_2+1)=r_1r_2+(r_1+r_2)+1=c-b+1=c-4=\frac{(c+1)}{2} \Rightarrow 2c-8=c+1 \Rightarrow c=9$

   Thus, $(b,c)=\underline{(5,9)}$.

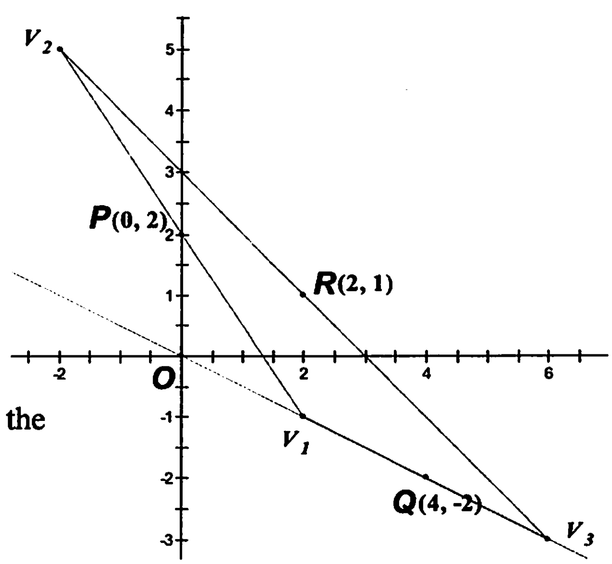
3. Let $P(0,2)$, $Q(4,-2)$ and $R(2,1)$ be the midpoints of $\overline{V_1V_2}$, $\overline{V_1V_3}$, and $\overline{V_2V_3}$, where the vertices of the triangle have coordinates $V_1(x_1,y_1)$, $V_2(x_2,y_2)$ and $V_3(x_3,y_3)$. Then:

   $\begin{cases} x_1+x_2=0 \\ x_2+x_3=4 \\ x_3+x_1=8 \end{cases} \Rightarrow (x_1,x_2,x_3)=(2,-2,6)$ and

   $\begin{cases} y_1+y_2=4 \\ y_2+y_3=2 \\ y_3+y_1=-4 \end{cases} \Rightarrow (y_1,y_2,y_3)=(-1,5,-3)$

   Thus, the vertices of the triangle are $V_1(2,-1)$, $V_2(-2,5)$ and $V_3(6,-3)$.

   From the diagram at the right, we see that V_1, V_3 and O are collinear, and that ΔV_2OV_3 will have the largest area.



To find the area of a triangle whose vertices are known (or any convex polygon for that matter), follow the following procedure:
List the coordinates of the vertices in CW (or CCW) order, repeating the starting vertex at the end of the list.
Along the downward diagonals, sum all possible products
Along the upward diagonals, sum all the possible products.
The required area is half the absolute value of the difference of these two sums.

Here are the calculations.
To zero out as many products as possible, we begin the list with the origin and proceed in the clockwise direction around the triangle.

0	0
-2	5
6	-3
0	0

Discounting all the zero products, the area is $\frac{1}{2}\left|(-2\cdot -3)-(6\cdot 5)\right|=\frac{1}{2}\left|6-30\right|=\underline{\mathbf{12}}$.
It is left to you to verify that this process works for any triangle, and for the truly ambitious, that it works for any convex polygon. I wonder: Does it also work for concave polygons? Have fun.

The following diagram and calculations verify our result

$$\mathcal{A}(V_2OV_3)=\mathcal{A}(I)+\mathcal{A}(II)+\mathcal{A}(III)$$
$$=\left(\frac{1}{2}\cdot 10-\frac{1}{2}\cdot 2^2\right)+\frac{1}{2}\cdot 3^2+\left(\frac{1}{2}\cdot 18-\frac{1}{2}\cdot 3^2\right)=3+9=\underline{\mathbf{12}}.$$