Greater Boston Mathematics League

G B M L

ANSWERS

1. ______________________

2. ______________________

3. ______________________

Meet 3 – December 11, 2019

The word "compute" calls for an exact answer in simplest form.

Round 1: Algebra 1 – Fractions and Word Problems

1. Solve: $\dfrac{x^2 - 7x + 12}{x - 3} = \dfrac{2x}{6 - x} - 3$

2. A jar is 60% full of a 15% salt solution. After 4 ounces spill out of the jar, a 10% salt solution is added until the jar is filled. Given that the jar ends up with a 12% salt solution, compute the capacity of the jar in ounces.

3. Tom and Polly are painting a fence. If Tom were to paint the fence alone, he would take 15 hours. If Polly were to paint the fence alone, she would take 16 hours. On Saturday, Tom and Polly paint different parts of the fence for 6 hours each. On Sunday, Jim comes and paints the rest of the fence, taking $2\frac{1}{4}$ hours. If the three painters had started painting together on Saturday, it would have taken them m minutes to complete the job. Compute the whole number nearest to m.

Greater Boston Mathematics League

G B M L

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

Meet 3 – December 11, 2019

The word "compute" calls for an exact answer in simplest form.

Round 2: Coordinate Geometry of the Straight Line

1. Let $f(x) = mx + 7$ for some positive m. If the distance between $(1, f(1))$ and $(3, f(3))$ is $2\sqrt{5}$, compute m.

2. The line $2x + 3y = 6$ is reflected in the line $x = 6$ to obtain line ℓ. Give the equation of ℓ in the form $Ax + By = C$ where A is positive, and A, B, C have no common factors other than 1.

3. Line ℓ passes through points $A(3j, 5j)$ and $B(-3k, -5k)$ where k is positive and $j + k = 4$. Line m passes through $(-4, 16)$, and is perpendicular to ℓ at point X. Given that $BX : XA = 3 : 1$, compute the least possible value of $j^2 + k^2$.

Greater Boston Mathematics League

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

Meet 3 – December 11, 2019

The word "compute" calls for an exact answer in simplest form.

Round 3: Geometry – Polygons: Area and Perimeter

1. In pentagon $MATHY$, $\overline{HM}$ divides the pentagon into square $MATH$ and equilateral triangle HYM. Given that the perimeter of $MATHY$ is 150, compute the area of $MATHY$.

2. Suppose that rhombus $RHOM$ is drawn with $RO = 12$ and $HM = 5$. Given that the perimeter of $RHOM$ is P and the area of $RHOM$ is A, compute $P + A$.

3. In trapezoid $GBML$, $\overline{BM} \parallel \overline{LG}$ and $\angle L$ is a right angle. Given that $m\angle BGL = 60°$, $\overline{MG}$ bisects $\angle BGL$, and the perimeter of $GBML$ is $42 + k\sqrt{3}$ for some integer k, compute k times the area of $GBML$.

Greater Boston Mathematics League

GBML

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

Meet 3 – December 11, 2019

The word "compute" calls for an exact answer in simplest form.

Round 4: Algebra 2 – Logarithms, Exponents, and Radicals

1. Suppose that $\log_{20} 3 = P$ and $\log_{20} 5 = Q$. The value of $\log_{20} 720 = aP + bQ + c$ for some integers a, b, and c. Compute the product abc.

2. Given that $\log_{12} 4 \approx 0.5579$, compute $\log_{12} 54$ *rounded to the nearest thousandth.*

3. Solve for x: $\sqrt{16x^4 + 9x^4} = \left(\frac{81}{25}\right)^{3/2} \cdot \left(\frac{1}{9}\right)^{-5/2}$

Greater Boston Mathematics League

G B M L

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

Meet 3 – December 11, 2019

The word "compute" calls for an exact answer in simplest form.

Round 5: Trigonometry – Analysis and Complex Numbers

1. Compute the least positive measure of an angle θ (in degrees) for which

$$\sin\theta^\circ = \cos^2 40^\circ - \sin^2 40^\circ.$$

2. Given that A and B are angles such that $\cos A = -\frac{4}{5}$, $\sin A > 0$, $\tan B = \frac{7}{24}$, and $\sin B < 0$, compute

$$\sin\left(\frac{\pi}{2} + A\right) \cdot \cos(B - \pi).$$

3. Suppose that z and w are complex numbers such that $z^3 = -108\sqrt{2} - 108i\sqrt{2}$ and $z \cdot w = -12i$. Given that $z = q(\cos\phi + i\sin\phi)$ for a third-quadrant angle ϕ and $w = r(\cos\theta + i\sin\theta)$ for θ measuring between 0° and 360°, compute the ordered pair (r, θ).

Greater Boston Mathematics League

G B M L

ANSWERS

(3 pts) 1. ______________________

(3 pts) 2. ______________________

(4 pts) 3. ______________________

Meet 3 – December 11, 2019

The word "compute" calls for an exact answer in simplest form.

Team Round

1. The polynomial $x^6 - x^4 - 16x^2 + 16$ is factored completely as a product of five binomials. Each of the leading coefficients of each of the binomials is $+1$. The sum of the five factors can be written in the form $Ax^2 + Bx + C$. Compute the sum in that form.

2. Triangle ABC has sides in the ratio $3 : 7 : 8$. The area of $\triangle ABC$ is $150\sqrt{3}$. Compute the perimeter of $\triangle ABC$.

3. Given that x is an angle for which $3\sin x + 5\cos x = \dfrac{23}{4}$, compute $8\cos(2x) + 15\sin(2x)$.

GBML Answer Sheet

Meet #3

Round 1.

1. 2

2. 20

3. 262

Round 2.

1. 2

2. $2x - 3y = 18$

3. 10

Round 3.

1. $900 + 225\sqrt{3}$

2. 56

3. $540\sqrt{3}$

Round 4.

1. -4

2. 1.605

3. $\pm\frac{243\sqrt{3}}{25}$

Round 5.

1. 10

2. $-\frac{96}{125}$

3. $(2, 75°)$

Team Round

1. $x^2 + 4x + 4$

2. 90

3. $\frac{257}{16}$

GBML Contest #3. SOLUTIONS

Round 1: Algebra 1 – Fractions and Word Problems

1. $\boxed{\mathbf{2}}$ The left-hand side of the equation is equivalent to $x-4$ as long as $x \neq 3$. Thus, $x-4=\frac{2x}{6-x}-3 \rightarrow x-1=\frac{2x}{6-x}$. Cross-multiplying, $-x^2-6+7x=2x \rightarrow 0=x^2-5x+6 \rightarrow 0=(x-2)(x-3)$. Because it was already established that $x \neq 3$, the answer is $x=\mathbf{2}$.

2. $\boxed{\mathbf{20}}$ Let the capacity of the jar be c ounces. The jar initially has $0.6 \cdot 0.15 \cdot c = 0.09c$ ounces of salt. Then, four ounces of a 15% salt solution spill out, leaving $0.09c - 0.60$ ounces of salt. The remaining capacity of the jar is $0.4c+4$ ounces, so solve $0.09c-0.60+0.10(0.4c+4)=0.12c$ to obtain $c=\mathbf{20}$ ounces.

3. $\boxed{\mathbf{262}}$ On Saturday, Tom painted $\frac{6}{15}=\frac{2}{5}$ of the fence and Polly painted $\frac{6}{16}=\frac{3}{8}$ of the fence, leaving $1-\frac{2}{5}-\frac{3}{8}=\frac{9}{40}$ of the fence to be painted by Jim. Jim did this in 135 minutes, so Jim would have painted $\frac{1}{40}$ of the fence in $135 \div 9 = 15$ minutes, so he would have painted the whole fence in $15 \cdot 40 \div 60 = 10$ hours. (Note also that 10 hours can be obtained by dividing the 9/4 hours that Jim paints by the 9/40 of the fence that needed to be painted.) Therefore, solve $\frac{m}{15}+\frac{m}{16}+\frac{m}{10}=1\cdot 60$ to obtain $m \approx \mathbf{262}$ minutes.

Round 2: Coordinate Geometry of the Straight Line

1. $\boxed{\mathbf{2}}$ Using the Pythagorean Theorem, $(3-1)^2+((3m+7)-(m+7))^2=(2\sqrt{5})^2$, which simplifies to obtain $4+4m^2=20$. This has one positive solution, $m=\mathbf{2}$.

2. $\boxed{\mathbf{2x-3y=18}}$ Substitute $x=6$ into $2x+3y=6$ and solve to obtain $y=-2$, so $(6,-2)$ is on the line ℓ. The x-intercept of $2x+3y=6$ is $(3,0)$, whose image after reflection in $x=6$ is $(9,0)$ which is on ℓ. The slope of ℓ is $\frac{-2-0}{6-9}=\frac{2}{3}$, so the equation of ℓ is $y=0=\frac{2}{3}(x-9) \rightarrow \mathbf{2x-3y=18}$.

3. $\boxed{\mathbf{10}}$ Points A and B lie on $y=\frac{5}{3}x$. Thus, the slope of m is $-\frac{3}{5}$. An equation of m is $y-16=-\frac{3}{5}(x+4) \rightarrow y=-\frac{3}{5}x+\frac{68}{5}$. Solve $\frac{-3x+68}{5}=\frac{5x}{3}$ to obtain $25x=-9x+204 \rightarrow x=6 \rightarrow y=10$. Now, solve the system $10-(-5k)=3(5j-10)$ and $j+k=4$ to obtain $j=3$ and $k=1$, so the value of j^2+k^2 is $3^2+1^2=\mathbf{10}$.

Suppose that A is instead between X and B. Then the problem statement implies $10+5k=3(10-5j)$ and $6+3k=3(3-3j)$, which implies $3j+k=4$. Also, $j+k=4$, so this system solves to obtain $j=0$ and $k=4$, and $j^2+k^2=16>10$.

Now suppose that B is between A and X. This implies that $AB+BX=AX$, but $BX:AX=3:1$ which means $BX=3AX \rightarrow AB+3AX=AX \rightarrow AB=-2AX$, and this is a contradiction. Therefore, the least possible value of j^2+k^2 is 10.

Round 3: Geometry – Polygons: Area and Perimeter

1. $\boxed{\mathbf{900+225\sqrt{3}}}$ Because the perimeter of $MATHY$ is 150, $HM = 150 \div 5 = 30$. The area of $MATH$ is $30^2 = 900$ and the area of HYM is $\dfrac{30^2\sqrt{3}}{4} = 225\sqrt{3}$, so the answer is $\mathbf{900+225\sqrt{3}}$.

2. $\boxed{\mathbf{56}}$ The perimeter of $RHOM$ is $4\cdot\sqrt{(12/2)^2+(5/2)^2} = 4\cdot\sqrt{\frac{12^2}{4}+\frac{5^2}{4}} = \dfrac{4}{2}\cdot\sqrt{12^2+5^2} = 26$. (Note that this calculation can be done speedily by recognizing a 5-12-13 Pythagorean triple.) The area of $RHOM$ is $\dfrac{1}{2}\cdot 12\cdot 5 = 30$. The answer is $P + A = 26 + 30 = \mathbf{56}$.

3. $\boxed{\mathbf{540\sqrt{3}}}$ Because $\overline{BM} \parallel \overline{LG}$ and $m\angle BGL = 60^\circ$ and $\overline{MG}$ bisects $\angle BGL$, $m\angle MGL = m\angle MGB = m\angle BMG = 30^\circ$. Suppose X is the foot of the perpendicular from B to $\overline{GL}$. Then $GB = BM = XL = 2GX$ and $ML = BX = GX\sqrt{3}$. Thus $7GX + GX\sqrt{3} = 42 + k\sqrt{3}$ so $GX = k = 6$. The area of $GBML$ is $\dfrac{1}{2}\cdot(12+18)\cdot 6\sqrt{3} = 90\sqrt{3}$. The answer is $6\cdot 90\sqrt{3} = \mathbf{540\sqrt{3}}$.

Round 4: Algebra 2 – Logarithms, Exponents, and Radicals

1. $\boxed{-4}$ Notice that $\log_{20} 720 = \log_{20}\left(\frac{400 \cdot 9}{5}\right) = 2 + 2P - Q$. Therefore, $abc = 2 \cdot -1 \cdot 2 = \mathbf{-4}$.

2. $\boxed{\mathbf{1.605}}$ First, $log_{12} 2 = \frac{1}{2}\log_{12} 4 \approx 0.2790$. Then $\log_{12} 3 = 1 - \log_{12} 4 \approx 0.4421$, which implies $\log_{12} 27 = 3\log_{12} 3 \approx 1.3263$. Then, $\log_{12} 54 \approx 1.3263 + 0.2790 \approx \mathbf{1.605}$.

3. $\boxed{\pm\frac{\mathbf{243\sqrt{3}}}{\mathbf{25}}}$ The given equation is equivalent to $\sqrt{25x^4} = \left(\frac{9}{5}\right)^3 (3)^5$, which implies $|5x^2| = \frac{3^6 \cdot 3^5}{5^3} = \frac{3^{11}}{5^3}$. This means that $x^2 = \frac{3^{11}}{5^4}$ and so $x = \pm\frac{\mathbf{243\sqrt{3}}}{\mathbf{25}}$.

Round 5: Trigonometry – Analysis and Complex Numbers

1. $\boxed{\mathbf{10}}$ The given equation is equivalent to $\sin\theta° = \cos(80°)$, so solve $\theta + 80 = 90$ to obtain $\theta = \mathbf{10}$.

2. $\boxed{-\frac{\mathbf{96}}{\mathbf{125}}}$ Draw right triangles with appropriate side lengths to obtain $\sin A = \frac{3}{5}$ and $\cos B = -\frac{24}{25}$. Then use identities to obtain $\sin\left(\frac{\pi}{2} + A\right) \cdot \cos(B - \pi) = \left(\sin\frac{\pi}{2}\cos A + \cos\frac{\pi}{2}\sin A\right)(\cos B\cos\pi + \sin B\sin\pi)$, which is $\left(1 \cdot -\frac{4}{5}\right)\left(-\frac{24}{25} \cdot -1\right) = -\frac{\mathbf{96}}{\mathbf{125}}$.

3. $\boxed{(\mathbf{2}, \mathbf{195°})}$ Because z^3 has polar coordinates $(216, 225°)$, DeMoivre's Theorem implies that z has polar coordinates $(\sqrt[3]{216}, 225° \div 3 + 120k°)$ or $(6, 75° + 120k°)$ for $k = 0$ or $k = 1$ or $k = 2$. Because ϕ is a third-quadrant angle, the polar coordinates for z are $(6, 195°)$. Because the product $z \cdot w$ has polar coordinates $(12, 270°)$, the polar coordinates of w are $(12 \div 6, 270° - 195°)$ or $(\mathbf{2}, \mathbf{75°})$.

Team Round

1. $\boxed{\boldsymbol{x^2+4x+4}}$ Recognize that
$x^6 - x^4 - 16x^2 + 16 = x^4(x^2-1) - 16(x^2-1) = (x^4-16)(x^2-1)$. Complete the factoring to obtain $(x-2)(x+2)(x^2+4)(x-1)(x+1)$. The sum of the factors is $\boldsymbol{x^2+4x+4}$.

2. $\boxed{\mathbf{90}}$ Let the three sides be $\boldsymbol{3k}$, $\boldsymbol{7k}$, and $\boldsymbol{8k}$. The smallest angle $\boldsymbol{\theta}$ lies opposite the side of length $\boldsymbol{3k}$. Using the Law of Cosines, $\boldsymbol{9k^2 = 49k^2 + 64k^2 - 112k^2\cos\theta}$. Since $\boldsymbol{k \neq 0}$,
$\boldsymbol{112\cos\theta = 104 \to \cos\theta = \dfrac{13}{14} \to \sin\theta = \dfrac{\sqrt{27}}{14}}$. The area of $\triangle \boldsymbol{ABC}$ is
$\boldsymbol{\frac{1}{2}(7k)(8k)\left(\dfrac{\sqrt{27}}{14}\right) = 150\sqrt{3}}$, which solves to obtain $\boldsymbol{k=5}$. The perimeter is $\boldsymbol{18(5) = 90}$.

-OR- Use Heron's Formula, where $\boldsymbol{s = \frac{3k+7k+8k}{2} = 9k}$. Then,
$\boldsymbol{150\sqrt{3} = \sqrt{9k(6k)(2k)(1k)} = 6k^2\sqrt{3}}$, so $\boldsymbol{6k^2 = 150}$, and $\boldsymbol{k=5}$. The perimeter is $\boldsymbol{18(5) = 90}$.

3. $\boxed{\boldsymbol{\dfrac{257}{16}}}$ Square both sides of the given equation to obtain
$\boldsymbol{9\sin^2 x + 25\cos^2 x + 30\sin x\cos x = \dfrac{529}{16}}$. This is equivalent to
$\boldsymbol{9\sin^2 x + 9\cos^2 x + 16\cos^2 x + 15\sin(2x)}$ or $\boldsymbol{9 + 16\cos^2 x - 8 + 8 + 15\sin(2x) = \dfrac{529}{16}}$.
This is equivalent to $\boldsymbol{8\cos(2x) + 15\sin(2x) = \dfrac{529}{16} - \dfrac{272}{16} = \dfrac{257}{16}}$. The answer is $\boldsymbol{\dfrac{257}{16}}$.

Author: George Reuter - coachreu@gmail.com

Reviewers: Mike Schockett - mschockett@comcast.net and Dick Olson - olson.re@gmail.com