Greater Boston Mathematics League

G B M L

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

Meet 5 – February 12, 2020

The word "compute" calls for an exact answer in simplest form.

Round 1: Arithmetic – Open

1. Eight of the counting numbers from 1 through 9 are to be written in the eight squares in the diagram below, one per square. The total of each row is given at the end of the row, and the total of each column is given at the bottom of the column. Compute the missing number.

□	□	□	**23**
□		□	**7**
□	□	□	**8**
15	**7**	**16**	

2. Compute the positive integer x for which the ordered pair of integers (x, y) satisfies $12xy + 4x + 3y = 2020$.

3. The number 2020 is a four-digit counting number which has two different repeated digits. How many counting numbers between 1000 and 9999 have two different repeated digits?

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Round 2: Algebra – Open

1. The product of three consecutive positive integers is 85 times the sum of those same three integers. Compute the mean of the three consecutive integers.

2. Let G, B, and M be three integers such that $(G+2M):(M+2B):(B+2G) = 1:2:3$ and $G+B+M = 2020$. Compute M.

3. Compute all values of x such that $x^3 + (2x-6)^3 = (3x-6)^3$.

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Round 3: Geometry – Open

1. Square $ABCD$ with side length 6 units is given. Points M and N are on $\overline{AB}$ and $\overline{CD}$ such that $MA = MB$ and $NC = 2ND$. Given that E is the midpoint of MN, compute $[ACE]$, the area of triangle ACE.

2. In rectangle $RECT$, J is the midpoint of $\overline{RT}$ and X is the midpoint of $\overline{CT}$. Given that $RX = 10$ and $CJ = 12$, compute RC.

3. Given tetrahedron $ABCD$ with $AB = BC = 10$, $AD = DC = 17$, $AC = 16$, and a right angle where planes ABC and ADC meet, compute the volume of $ABCD$.

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Round 4: Algebra 2 – Open

1. The domain of $f(x) = \sqrt{20 - \sqrt{2x+1}}$ is the set of all x such that $a \le x \le b$. Compute $a + b$.

2. Compute the value of N for which $\dfrac{1}{\log_{16} N} + \dfrac{1}{\log_{81} N} = 2$.

3. Consider a sequence $\{a_n\}$ for which $a_1 = 2$, $a_2 = 0$, $a_3 = 20$, $a_4 = 2020$, and $a_i = a_{i-1} - a_{i-2} + a_{i-3} - a_{i-4}$ for $i \ge 5$. Compute a_{2020}.

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2. ______________________________

3. ______________________________

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Round 5: PreCalculus – Open

1. Compute the constant term in the expansion of $\left(x + \dfrac{3}{x^2}\right)^9$.

2. The roots of $243x^3 - 243x^2 + Bx - 5 = 0$ are in arithmetic progression. Compute the integer B.

3. Given that $i = \sqrt{-1}$, the equation $2z^3 = (-\sqrt{3} + i)^4$ has three complex solutions. Each of these solutions can be written in the form $r(\cos\theta^\circ + i\sin\theta^\circ)$ where $r > 0$ and $0 \le \theta < 360$. Compute $r + \theta_1 + \theta_2 + \theta_3$, where the θ_i's are the values of θ for the three solutions.

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(3 pts) 1. ________________________

(3 pts) 2. ________________________

(4 pts) 3. ________________________

Meet 5 – February 12, 2020

The word "compute" calls for an exact answer in simplest form.

Team Round

1. Gene Beamel makes the letter "W" by connecting the points $(-6,7)$, $(-2,-5)$, $(0,1)$, $(2,-5)$, and $(6,7)$ in order with line segments. Those four segments are parts of the graph of $y = ||ax| - b| - c$ where a, b, and c are positive integers. Compute the ordered triple (a,b,c).

2. Compute the number of four-digit base-ten numbers that have distinct nonzero digits and are divisible by 99.

3. In pentagon $ABCDE$, $AB = AE$, $BC = ED$, $CD = 20$, $\angle BCD$ and $\angle EDC$ are right angles, and all side lengths are integers. Given that the area of $ABCDE$ is 2020, compute the perimeter of $ABCDE$.

GBML Answer Sheet

Meet #5

Round 1.

1. 7

2. 505

3. 459

Round 2.

1. 16

2. 0

3. $0, 2, 3$ *[must have all three]*

Round 3.

1. $\dfrac{3}{2}$ or $1\dfrac{1}{2}$ or 1.5

2. $\frac{4}{5}\sqrt{305}$

3. 240

Round 4.

1. 199

2. 36

3. -1998

Round 5.

1. 2268

2. 69

3. 602

Team Round

1. $(3, 6, 5)$

2. 48

3. 250

GBML Contest #5. SOLUTIONS

Round 1: Arithmetic – Open

1. $\boxed{\mathbf{7}}$ The top row adds to 23. Since the numbers in the figure are distinct, the three numbers must include 9. Further, the only way to have two numbers that add to 14 are 6 and 8 (since 9 has already been used). Notice that 7 cannot be used in the second row (since the row adds to 7) or in the third row (the row adds to 8). Therefore, the missing number is **7**.

Alternate Solution: Find the sum of the row sums or the column sums to find that the sum of the eight numbers is $23 + 7 + 8 = 15 + 7 + 16 = 38$. The sum of the integers from 1 through 9 is $9(10)/2 = 45$. Thus, the missing number is $45 - 38 = 7$.

2. $\boxed{\mathbf{505}}$ Add 1 to both sides to obtain $(4x + 1)(3y + 1) = 2021 = 43 \cdot 47 = 1 \cdot 2021$. If $4x + 1 = 43$ or $4x + 1 = 47$, then x is not an integer. If $4x + 1 = 1$, then $x = 0$ is not a positive integer. Thus, $4x + 1 = 2021$ and $x = \mathbf{505}$.

Notice that the wording of the question implies that there is only one ordered pair (x, y) of integers $(x > 0)$ for which the given equation holds. By inspection, $(505, 0)$ is a solution. Thus, it must be the only solution.

3. $\boxed{\mathbf{459}}$ First, consider numbers which don't have zero as a digit. These numbers are of the form $\underline{A}\,\underline{A}\,\underline{B}\,\underline{B}$ where $A \neq B \neq 0$ or some rearrangement thereof. There are 9 choices for A and 8 choices for B, so there are $9 \cdot 8 = 72$ choices for (A, B). There are $\binom{4}{2} = 6$ ways to arrange the digits of $\underline{A}\,\underline{A}\,\underline{B}\,\underline{B}$ to form a four-digit number between 1000 and 9999, so there are $72 \cdot 6 = 432$ four-digit numbers of the required form that do not have 0 as a digit.
Now, consider numbers which have 0 as a digit. These numbers are of the form $\underline{C}\,\underline{C}\,\underline{0}\,\underline{0}$ where $C \neq 0$ or some rearrangement thereof. There are 9 choices for C. There are $\binom{3}{1} = 3$ ways to arrange the digits of $\underline{C}\,\underline{C}\,\underline{0}\,\underline{0}$ to form a four-digit number between 1000 and 9999 because there are 3 places for the second C, so there are $9 \cdot 3 = 27$ four-digit numbers of the required form that have 0 as a digit.
The number of counting numbers between 1000 and 9999 that have two different repeated digits is $432 + 27 = \mathbf{459}$.

Round 2: Algebra – Open

1. $\boxed{\mathbf{16}}$ Let the integers be $x-1$, x, and $x+1$. Then their mean is x. Solve $(x-1)(x)(x+1) = 85(3x)$ to obtain $x^2 - 1 = 255 \rightarrow x = \mathbf{16}$.

2. $\boxed{\mathbf{0}}$ Let $G + 2M = k$, $M + 2B = 2k$, and $B + 2G = 3k$. By adding equal quantities, it follows that $3(G + B + M) = 6k \rightarrow G + B + M = 2k$. Substituting $G + B + M = 2020$ yields $k = 1010$. Therefore, $2B + M = 2020$. Subtract to obtain
$G + B + M - (2B + M) = 2020 - 2020 \rightarrow G - B = 0$ so $G = B$, which implies
$B + 2G = B + 2B = 3B = 3030 \rightarrow B = 1010 = G$, so $M = 2020 - 2020 = \mathbf{0}$.

Alternate Solution: Because $\dfrac{G+2M}{M+2B} = \dfrac{1}{2}$, it follows that $2G + 4M = M + 2B \rightarrow 2G + 3M = 2B$.
Also, because $\dfrac{G+2M}{B+2G} = \dfrac{1}{3}$, it follows that
$3G + 6M = B + 2G \rightarrow G + 6M = B \rightarrow 2G + 12M = 2B$. Subtracting equal quantities,
$2G + 12M - (2G + 3M) = 2B - 2B$, which implies $9M = 0$ and so $M = 0$.

3. $\boxed{\mathbf{0, 2, 3}}$ The equation is of the form $A^3 + B^3 = (A + B)^3$, which has a solution only if $A = 0$ or $B = 0$ or $A + B = 0$. Solve $x = 0$ and $2x - 6 = 0$ and $3x - 6 = 0$ to obtain the solutions: $\mathbf{0, 2, 3}$.

Round 3: Geometry – Open

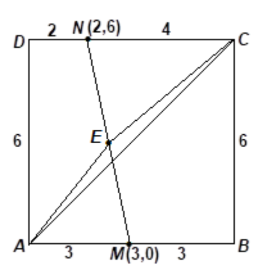
1. $\boxed{\dfrac{\mathbf{3}}{\mathbf{2}}}$ Note that E must lie inside $\triangle ADC$, because if it did not, then NC/ND would only be at most 1. Therefore, we can use complementary counting, i.e.,

$$[ACE] = [ADC] - [AED] - [CED].$$

The area of triangle ADC is $\dfrac{6 \cdot 6}{2} = 18$. However, the areas of triangles AED and CED are a bit more complicated. It is given that N lies on $\overleftrightarrow{CD}$, and M lies 6 units away from $\overleftrightarrow{CD}$, so E must lie $6/2 = 3$ units away from $\overleftrightarrow{CD}$. Therefore, $[CED] = \dfrac{6 \cdot 3}{2} = 9$. Additionally, N lies $6 \cdot (1/3) = 2$ units away from $\overleftrightarrow{AD}$, and M lies $6 \cdot (1/2) = 3$ units away from $\overleftrightarrow{AD}$ on the same side, so E must lie $\frac{2+3}{2} = 2.5$ units away from $\overleftrightarrow{AD}$. Therefore, $[ADE] = \frac{6 \cdot 2.5}{2} = 15/2$. This gives the area of triangle ACE:

$$[ACE] = 18 - 9 - (15/2) = \frac{\mathbf{3}}{\mathbf{2}}.$$

Alternate Solution: Place $ABCD$ in the coordinate grid so that $A(0,0)$ and $D(0,6)$. Then, $C(6,6)$ and $E(2.5,3)$ as shown below.



Using the coordinates of the vertices, $[ACE] = \frac{1}{2}\left\|\begin{vmatrix} 0 & 0 & 1 \\ 2.5 & 3 & 1 \\ 6 & 6 & 1 \end{vmatrix}\right\|$. Expanding by minors, this area is

$$\frac{1}{2}\left|0 \cdot (+1) \cdot \begin{vmatrix} 3 & 1 \\ 6 & 1 \end{vmatrix} + 0 \cdot (-1) \cdot \begin{vmatrix} 2.5 & 1 \\ 6 & 1 \end{vmatrix} + 1 \cdot (+1) \cdot \begin{vmatrix} 2.5 & 3 \\ 6 & 6 \end{vmatrix}\right|,$$

which is $\frac{1}{2}|15 - 18| = 1.5$. The placement of one vertex at the origin makes the calculations particularly easy.

An alternative strategy (which works for ANY polygon - convex or concave) goes as follows. In a vertical column, list the coordinates of the vertices in a clockwise or counterclockwise order, starting at any vertex. Repeat the coordinates of the point with which you started. Subtract the sum of the UPWARD diagonal products from the sum of the DOWNWARD diagonal products, and take half the absolute value of this difference. For practice, apply the strategy to confirm that the area of the nondescript convex quadrilateral $QUAD$ with vertices $Q(-3, 3)$, $U(4, -4)$, $A(7, 7)$, and $D(-16, -2)$, is 50 square units regardless of the vertex with which you start the column. Can you verify this area using an entirely different strategy?

2. $\boxed{\mathbf{\frac{4}{5}\sqrt{305}}}$ Let $JT = a$ and $TX = b$. Applying the Pythagorean Theorem to $\triangle RTX$, $(2a)^2 + b^2 = 10^2$. Applying the Pythagorean Theorem to $\triangle JTC$, $a^2 + (2b)^2 = 12^2$. This implies $5a^2 + 5b^2 = 100 + 144 = 244$. Applying the Pythagorean Theorem to $\triangle RTC$, $RC^2 = 4a^2 + 4b^2 = 244 \cdot \frac{4}{5} = 4 \cdot \frac{244}{5}$. Therefore, $RC = 2\sqrt{\frac{244}{5}} = 4\sqrt{\frac{61}{5}} = \mathbf{\frac{4}{5}\sqrt{305}}$.

3. $\boxed{\mathbf{240}}$ Orient the tetrahedron such that $\triangle ACD$ is "on the floor" and $\triangle ABC$ is perpendicular to the "floor". The area of $\triangle ACD$ is $\frac{1}{2} \cdot 16 \cdot \sqrt{17^2 - 8^2} = 120$. Now, let $\overline{BX}$ be the altitude from B to $\triangle ACD$. It follows that $BX = \sqrt{10^2 - 8^2} = 6$, so the volume of $ABCD$ is $\frac{1}{3} \cdot 120 \cdot 6 = \mathbf{240}$.

Round 4: Algebra 2 – Open

1. $\boxed{\mathbf{199}}$ Because $2x + 1 \geq 0$, $x \geq -\frac{1}{2}$. Because $20 - \sqrt{2x+1} \geq 0$, $2x + 1 \leq 400 \to x \leq \frac{399}{2}$. Therefore, $a + b = \frac{-1}{2} + \frac{399}{2} = \frac{398}{2} = \mathbf{199}$.

2. $\boxed{\mathbf{36}}$ Applying the change-of-base rule, the given equation is equivalent to $\frac{\log 16}{\log N} + \frac{\log 81}{\log N} = \frac{\log(2 \cdot 3)^4}{\log N} = 2$. Therefore, $6^4 = N^2$, so $N = 6^2 = \mathbf{36}$.

3. $\boxed{\mathbf{-1998}}$ Compute the next few terms: $a_5 = 2020 - 20 + 0 - 2 = 1998$, $a_6 = 1998 - 2020 + 20 - 0 = -2 = -a_1$, $a_7 = -2 - 1998 + 2020 - 20 = 0 = -a_2$, $a_8 = 0 - (-2) + 1998 - 2020 = -20 = -a_3$, $a_9 = -20 - 0 + (-2) - 1998 = -2020 = -a_4$, $a_{10} = -2020 - (-20) + 0 - (-2) = -1998 = -a_5$, and this pattern repeats every ten terms. Therefore, $a_{2020} = a_{10} = \mathbf{-1998}$.

Round 5: PreCalculus – Open

1. $\boxed{\mathbf{2268}}$ The constant term must have twice as large a power of x as it does of $\frac{1}{x^2}$. Therefore, the power of x must be 6 and the power of $\frac{1}{x^2}$ must be 3. Thus, the constant term is $\binom{9}{3}x^6\left(\frac{3}{x^2}\right)^3 = \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}\cdot 27 = 3\cdot 4\cdot 7\cdot 27 = 28\cdot 81 = \mathbf{2268}$.

2. $\boxed{\mathbf{69}}$ Let the roots be $q-d$, q, and $q+d$, so from Viete's formulas, the sum of the roots of this cubic equation is $1 = 3q$, so one root q is $\frac{1}{3}$. The product of the roots is $\frac{5}{243}$ by Viete's formulas, and this product is $q(q^2-d^2)$, so solve $1\left(1-d^2\right) = \frac{5}{243}$ to obtain $d = \frac{2}{9}$. Therefore, the three roots are $\frac{1}{9}$, $\frac{1}{3}$, and $\frac{5}{9}$. The value of B is related to the sum of the pairwise products, which is $\frac{1}{9}\cdot\frac{1}{3}+\frac{1}{9}\cdot\frac{5}{9}+\frac{1}{3}\cdot\frac{5}{9} = \frac{23}{81}$, which is $\frac{B}{243}$. The value of B is $\mathbf{69}$.

Alternate Solution: As in the Solution, determine that $\frac{1}{3}$ is a root. Then, use synthetic substitution to find that $\frac{1}{3}(B-54)-5=0$, which solves to obtain $B=69$.

3. $\boxed{\mathbf{602}}$ Notice that 2 corresponds to the polar point $(2,0)$ and $(-\sqrt{3}+i)$ corresponds to the polar point $(2,150^\circ)$. Thus, $(-\sqrt{3}+i)^4$ corresponds to the polar point $(2^4, 4\cdot 150^\circ) = (16, 600^\circ) = (16, 240^\circ)$ by DeMoivre's Theorem. Therefore, z^3 corresponds to the polar point $(8, 240^\circ)$ and z can be represented by the polar points $(2, 80^\circ)$ or $(2, 200^\circ)$ or $(2, 320^\circ)$. Thus, the final answer is $2+80+200+320 = \mathbf{602}$.

Team Round

1. $\boxed{(\mathbf{3},\mathbf{6},\mathbf{5})}$ Because the graph is symmetric with respect to the y-axis, consider the part of the graph for which $x \geq 0$. The slope of the segment from $(2,-5)$ to $(6,7)$ is $\frac{7-(-5)}{6-2} = 3$, so $a = 3$. The minimum y-value of any point on the graph is -5, so $c = 5$. Now, substitute $(0,1)$ for (x,y) and solve to obtain $1 = ||3(0)| - b| - 5 \to b = 6$. The desired ordered triple is $(\mathbf{3},\mathbf{6},\mathbf{5})$.

2. $\boxed{\mathbf{48}}$ Suppose that $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ is divisible by 99. Because $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ is divisible by 11, it is true that either (i) $A - B + C - D = 0 \to A + C = B + D$ or (ii) $A - B + C - D = 11 \to A + C = B + D + 11$ or (iii) $A - B + C - D = -11 \to A + C + 11 = B + D$. Because $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ is divisible by 9 and the digits are all distinct and nonzero, it is true that (a) $A + B + C + D = 18$ or (b) $A + B + C + D = 27$. Six cases result from this.

Case 1: Suppose (i) and (a) are true. This implies $A + C = 9$ and $B + D = 9$. For every choice of A, there is 1 choice for C and 6 choices for the (B, D) pair, so this case produces $8 \cdot 1 \cdot 6 = 48$ possible base-ten numbers.

Case 2: Suppose (i) and (b) are true. This is impossible because $A + C = 27/2$ has no solutions among the integers.

Case 3: Suppose (ii) and (a) are true. This is impossible because it implies $2B + 2D = 7$ and 7 is not even.

Case 4: Suppose (ii) and (b) are true. This implies $B + D = 8$ and $A + C = 19$, the latter of which is impossible.

Case 5: Suppose (iii) and (a) are true. In a manner similar to Case 3, this is impossible.

Case 6: Suppose (iii) and (b) are true. In a manner similar to Case 4, this is impossible.

Thus, the answer is **48**.

Alternate Solution: Let the four-digit integer be $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$. Then $A + B + C + D = 9$ or 18 or 27. Proceed by cases.

Case 1: Suppose $A + C = B + D$. This is possible only if $A + B + C + D = 18$ so $A + C = B + D = 9$, so (A, C) and (B, D) are each one of $(1, 8)$, $(2, 7)$, $(3, 6)$, or $(4, 5)$, and these can be combined in $\binom{4}{2} = 6$ ways. Each of these ways gives rise to 8 different four-digit numbers because if the chosen ordered pairs are $(i, 9 - i)$ and $(j, 9 - j)$, then there are 4 choices for A (which uniquely determines C) and 2 choices for B (which uniquely determines D). Thus, there are $6 \cdot 8 = 48$ possible numbers of this type.

Case 2: Suppose $A + C = 11 + B + D$ without loss of generality. Then, $(A + C, B + D) = (17, 6)$ or $(16, 5)$ or $(15, 4)$ or $(14, 3)$. None of these generate a value of $A + B + C + D$ that is divisible by 9. Thus, there are no possible numbers of this type.

The answer is 48.

3. $\boxed{\mathbf{250}}$ Drop a segment from A to $\overline{CD}$ with foot X such that $\overline{AX} \perp \overline{CD}$; let $\overline{AX}$ intersect $\overline{BE}$ at Y. Then, the area of $ABCDE$ is $2020 = 20 \cdot BC + 10 \cdot AY$. This implies $BC + \dfrac{AY}{2} = 101$, so $AY = 202 - 2BC$ is an integer. Because AB is also an integer, it must be that $(10, AY, AB)$ is a Pythagorean triple, and there is only one triple with 10 as the length of a leg, namely the $(10, 24, 26)$ triple. Thus, $AB = AE = 26$, and $AY = 24 = 202 - 2BC \to BC = ED = 89$. Then, the perimeter of $ABCDE$ is $26 + 26 + 89 + 89 + 20 = \mathbf{250}$.

Author: George Reuter - coachreu@gmail.com
Reviewers: Mike Schockett - mschockett@comcast.net and Dick Olson - olson.re@gmail.com