Greater Boston Mathematics League

G B M L

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

Meet 4 – January 15, 2020

The word "compute" calls for an exact answer in simplest form.

Round 1: Volume and Surface Area of Solids

1. The edges of a rectangular solid have lengths in the ratio $1:3:5$. The volume of the solid is 120 cubic cm. Compute the surface area of the solid in square cm.

2. A cube is inscribed in a sphere of radius 6 cm. Compute the volume of the space inside the sphere, but outside the cube, in cubic cm.

3. A right prism has bases that are regular hexagons. Given that the height of the prism has integer length, and the surface area of the prism is $(300\sqrt{3}+300)$ square cm, compute the volume of the prism in cubic cm.

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Round 2: Algebra – Inequalities and Absolute Value

1. Compute the number of integers n that satisfy $20.20 \leq 4n + 5 \leq 2020$.

2. Solve for x: $|x| - 4 + |x + 4| = 2x + 4$.

3. There are four values of x that satisfy $|\sqrt{9x^2 - 12x + 4} - 16| = 6$. Compute the sum of the squares of these four values.

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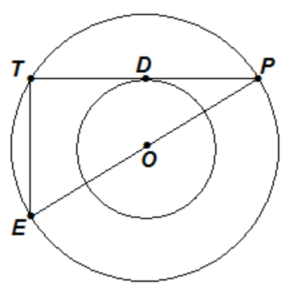
Round 3: Geometry – Similar Polygons, Circles, Area

1. Two circles O_1 and O_2 have radii R and r respectively. If $R - r = 4$ and the area of O_1 is 4 times the area of O_2, compute r.

2. In trapezoid $GBML$, $\overline{GB} \parallel \overline{ML}$, $\overline{GM}$ intersects $\overline{BL}$ at X, $GX : MX = 3 : 5$, and the area of $GBML$ is 256. Compute the area of $\triangle BMX$.

3. Two concentric circles with center O are drawn. The radius of the smaller circle is 3. A diameter $\overline{PE}$ of the larger circle is drawn, and segment $\overline{PT}$ is tangent to the smaller circle at D.



Given that the area of $\triangle PET$ is 24, compute the radius of the larger circle.

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Round 4: Algebra 2 – Sequences and Complex Numbers

1. Given that $i = \sqrt{-1}$, compute the value of $\dfrac{1 + i + i^2 + \cdots + i^{20}}{1 + i + i^2 + \cdots + i^{17}}$ in the form $A + Bi$, where A and B are real numbers.

2. Let $i = \sqrt{-1}$. In a geometric sequence $\{a_n\}$, $a_1 = 2 + i$ and $a_2 = 5$. Compute a_4 in simplest $a + bi$ form.

3. For a given sequence $\{b_n\}$, b_1 and b_2 are integers and $b_{n+2} = b_{n+1} - 3b_n$ for all positive integers n. Given that $b_4 + b_8 = -175$ and $b_5 + b_7 = -13$, compute $b_1^3 + b_2^3$.

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Meet 4 – January 15, 2020

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Round 5: Advanced – Conics

1. Compute the coordinates of the focus of the parabola with equation $x^2 - 20x + 20y + 2020 = 0$.

2. Circle C has equation $x^2 - 6x + y^2 - 4y = 36$. Line ℓ is drawn from $P(-5, -4)$ tangent to circle C at point X. Compute PX.

3. The ellipse with equation $\dfrac{(x-1)^2}{49} + \dfrac{(y-2)^2}{33} = 1$ has vertices V_1 and V_2 and foci F_1 and F_2. A hyperbola has vertices F_1 and F_2 and foci V_1 and V_2. Compute the y-coordinate of the fourth-quadrant point on the graph of the hyperbola for which the x-coordinate is 9.

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(3 pts) **1.** ______________________________

(3 pts) **2.** ______________________________

(4 pts) **3.** ______________________________

Meet 4 – January 15, 2020

The word "compute" calls for an exact answer in simplest form.

Team Round

1. A positive integer is chosen at random from the set of positive integer factors of 2020. Compute the probability that the chosen factor is less than 20.

2. Gene Beamel chooses 9 integers at random without replacement from the set of 14 integers $\{1, 2, 3, \cdots, 13, 14\}$. Compute the probability that the median of Gene's 9 integers is 7.

3. Jan chooses a set of four letters at random from the letters in the word PERPENDICULAR. Compute the number of distinct ways Jan may do this. *Note: the set* $\{D, E, E, R\}$ *is not distinct from the set* $\{R, E, E, D\}$.

GBML Answer Sheet

Meet #4

Round 1.

1. 184

2. $288\pi - 192\sqrt{3}$ or $96(3\pi - 2\sqrt{3})$

3. $750\sqrt{3}$

Round 2.

1. 500

2. -2

3. $\frac{1184}{9}$

Round 3.

1. 4

2. 60

3. 5

Round 4.

1. $\frac{1}{2} - \frac{1}{2}i$

2. $15 - 20i$

3. 19

Round 5.

1. $(10, -101)$

2. $\sqrt{51}$

3. $2 - 3\sqrt{11}$

Team Round

1. $\frac{5}{12}$

2. $\frac{75}{286}$

3. 321

GBML Contest #4. SOLUTIONS

Round 1: Volume and Surface Area

1. $\boxed{\mathbf{184}}$ The edges have lengths x, $3x$, and $5x$, so the volume is $15x^3 = 120 \to x = 2$. The surface area is $2 \cdot x \cdot 3x + 2 \cdot x \cdot 5x + 2 \cdot 3x \cdot 5x = 46x^2$, or $46 \cdot 2^2 = \mathbf{184}$ square cm.

2. $\boxed{\mathbf{288\pi - 192\sqrt{3}} \text{ or } \mathbf{96(3\pi - 2\sqrt{3})}}$ The volume of the sphere is $\frac{4}{3}\pi(6)^3 = 288\pi$ cubic cm. A diameter of the sphere is a space diagonal of the cube, so the side length s of the cube is found by solving $s\sqrt{3} = 12 \to s = 4\sqrt{3}$ cm. The volume of the cube is thus $(4\sqrt{3})^3 = 192\sqrt{3}$ cubic cm. The answer is $(\mathbf{288\pi - 192\sqrt{3}})$ cubic cm.

3. $\boxed{\mathbf{750\sqrt{3}}}$ Let the side length of one of the hexagons be s and the height of the prism be h. Then the surface area of a right regular hexagonal prism is $2 \cdot \dfrac{6s^2\sqrt{3}}{4} + 6sh$, or $3s^2\sqrt{3} + 6sh$. This surface area is equal to $(300\sqrt{3} + 300)$, and therefore, $6sh = 300$ or $6sh = 300\sqrt{3}$. Suppose $6sh = 300\sqrt{3}$; then $3s^2\sqrt{3} = 300$, which implies $s = \dfrac{10}{3^{1/4}}$ and $h = 5 \cdot 3^{3/4}$, which is not an integer, so this is a contradiction. As a result, $6sh = 300$, and $3s^2 = 300 \to s = 10 \to h = 5$. Now the volume of the prism is $\dfrac{6s^2\sqrt{3}}{4} \cdot h = \dfrac{6 \cdot 100\sqrt{3}}{4} \cdot 5 = \mathbf{750\sqrt{3}}$ cubic cm.

Round 2: Inequalities and Absolute Value

1. $\boxed{\mathbf{500}}$ By algebra, the given inequality is equivalent to $15.20 \le 4n \le 2015 \to 3.8 \le n \le 503.75$. The integers that satisfy this are $\{4, 5, 6, \cdots, 503\}$, and there are $503 - 4 + 1 = \mathbf{500}$ of them.

2. $\boxed{\mathbf{-2}}$ There are three cases: $x \ge 0$, $-4 < x < 0$, and $x \le -4$.
If $x \ge 0$, the given equation is equivalent to $x - 4 + x + 4 = 2x + 4$, which has no solution.
If $-4 < x < 0$, the given equation is equivalent to $-x - 4 + x + 4 = 2x + 4 \to x = -2$.
If $x \le -4$, the given equation is equivalent to $-x - 4 - x - 4 = 2x + 4 \to -12 = 4x \to x = -3$, which does not satisfy $x \le -4$.
The solution is $\mathbf{-2}$.

3. $\boxed{\mathbf{\frac{1184}{9}} \text{ or } \mathbf{131\frac{5}{9}}}$ Rewrite $|\sqrt{9x^2 - 12x + 4} - 16| = 6$ as $|\sqrt{(3x-2)^2} - 16| = 6 \to ||3x - 2| - 16| = 6$. Therefore, $|3x - 2| - 16 = \pm 6$ so $|3x - 2| = 22$ or $|3x - 2| = 10$. The first of these latter absolute value equations results in $3x - 2 = 22 \to x = 8$ or $3x - 2 = -22 \to x = \frac{-20}{3}$. The second of the latter absolute value equations results in $3x - 2 = 10 \to x = 4$ or $3x - 2 = -10 \to x = \frac{-8}{3}$. The sum of the squares of these four values is $\dfrac{400}{9} + \dfrac{64}{9} + 16 + 64 = \mathbf{\dfrac{1184}{9}}$.

Round 3: Geometry – Areas and Volumes

1. $\boxed{4}$ The problem situation is equivalent to $\pi(r+4)^2 = 4\pi r^2$, which implies $4r^2 = r^2 + 8r + 16 \to 3r^2 - 8r - 16 = 0$. Factoring yields $(3r+4)(r-4) = 0$, and the positive solution to this equation is **4**. Alternatively, the fact that the area of O_1 is 4 times the area of O_2 implies $R = 2r$, so $2r - r = 4 \to r = 4$.

2. $\boxed{\mathbf{60}}$ Because $GX : MX = 3 : 5$, it follows that the areas of $\triangle GXB$ and $\triangle MXL$ are in the ratio $9 : 25$, so these are $9k$ and $25k$ for some k. Also, the areas of $\triangle GXB$ and $\triangle BMX$ are in the ratio $3 : 5$, so the area of $\triangle BMX$ is $15k$. Similarly, the area of $\triangle GXL$ is $15k$. Solve $9k + 15k + 25k + 15k = 256$ to obtain $k = 4$ and thus the area of $\triangle BMX$ is $15 \cdot 4 = \mathbf{60}$.

3. $\boxed{5}$ Draw in segment OD. It follows that $\triangle DPO \sim \triangle TPE$, so $OD = 3$ and $TE = 6$. It is also true that $\triangle PET$ is right because $\angle PTE$ is inscribed on a diameter. Because the area of $\triangle PET$ is 24, it follows that $\frac{1}{2} \cdot 6 \cdot TP = 24 \to TP = 8$. Now, $\triangle PET$ is a $6-8-10$ right triangle, so $PE = 10$ and therefore the radius of the larger circle is $10 \div 2 = \mathbf{5}$.

Round 4: Algebra 2 – Sequences and Complex Numbers

1. $\boxed{\mathbf{\frac{1}{2} - \frac{1}{2}i}}$ Recognize that $i^k + i^{k+1} + i^{k+2} + i^{k+3} = 0$ for all integers k, so the desired quotient is $\frac{1}{1+i} = \frac{1-i}{1-(-1)}$, or $\mathbf{\frac{1}{2} - \frac{1}{2}i}$.

2. $\boxed{\mathbf{15 - 20\boldsymbol{i}}}$ First, find $r = \frac{a_2}{a_1} = \frac{5}{2+i}$. Rationalize the denominator to obtain $r = \frac{5(2-i)}{(2+i)(2-i)} = \frac{10-5i}{5} = 2 - i$. Then, $a_4 = a_2 \cdot r^2 = 5 \cdot (2-i)^2 = 5(3-4i) = \mathbf{15 - 20\boldsymbol{i}}$.

3. $\boxed{19}$ Find $\boldsymbol{b_3, b_4, b_5, b_6, b_7, b_8}$ in terms of $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$. The algebra is shown below.

$\boldsymbol{b_3 = b_2 - 3b_1}$

$\boldsymbol{b_4 = b_3 - 3b_2 = (b_2 - 3b_1) - 3(b_2) = -2b_2 - 3b_1}$

$\boldsymbol{b_5 = b_4 - 3b_3 = (-2b_2 - 3b_1) - 3(b_2 - 3b_1) = -5b_2 + 6b_1}$

$\boldsymbol{b_6 = b_5 - 3b_4 = (-5b_2 + 6b_1) - 3(-2b_2 - 3b_1) = b_2 + 15b_1}$

$\boldsymbol{b_7 = b_6 - 3b_5 = (b_2 + 15b_1) - 3(-5b_2 + 6b_1) = 16b_2 - 3b_1}$

$\boldsymbol{b_8 = b_7 - 3b_6 = (16b_2 - 3b_1) - 3(b_2 + 15b_1) = 13b_2 - 48b_1}$

The problem statement implies $\boldsymbol{11b_2 - 51b_1 = -175}$ and $\boldsymbol{11b_2 + 3b_1 = -13}$. Subtract equal quantities to obtain $\boldsymbol{-54b_1 = -162}$ and so $\boldsymbol{b_1 = 3}$. Substituting and solving, $\boldsymbol{b_2 = -2}$, and so the final answer is $\boldsymbol{3^3 + (-2)^3 = 27 - 8 = 19}$.

Round 5: Advanced – Conics

1. $\boxed{\mathbf{(10, -101)}}$ Convert the given equation to the form $(x-h)^2 = 4p(y-k)$. By algebra, the given equation is equivalent to $x^2 - 20x + 100 = -20y - 2020 + 100 \rightarrow (x-10)^2 = -20(y+96)$. The vertex is at $(10, -96)$, and $p = -20/4 = -5$, so the focus is at $(\mathbf{10}, \mathbf{-101})$.

2. $\boxed{\sqrt{\mathbf{51}}}$ Find the center and radius of circle C by rewriting the equation as $x^2 - 6x + 9 + y^2 - 4y + 4 = 36 + 9 + 4 \rightarrow (x-3)^2 + (y-2)^2 = 49$. Thus the radius of circle C is $\sqrt{49} = 7$ and the distance from P to the center of circle C is $\sqrt{8^2 + 6^2} = 10$. Therefore, $PX = \sqrt{10^2 - 7^2} = \sqrt{\mathbf{51}}$.

3. $\boxed{\mathbf{2 - 3\sqrt{11}}}$ The center of the ellipse (and of the hyperbola) is $(1, 2)$. The vertices V_1 and V_2 are at $(1 \pm 7, 2)$. The distance from the center to a focus is $\sqrt{49 - 33} = 4$, so the foci F_1 and F_2 are at $(1 \pm 4, 2)$. For the hyperbola, it is known that $c = 7$ and $a = 4$, so $b = \sqrt{33}$, and an equation of the hyperbola is $\dfrac{(x-1)^2}{16} - \dfrac{(y-2)^2}{33} = 1$. The y-coordinate of the fourth-quadrant point on the graph of the hyperbola for which the x-coordinate is 9 is obtained by solving $\dfrac{8^2}{16} - \dfrac{(y-2)^2}{33} = 1 \rightarrow \dfrac{(y-2)^2}{33} = 3$, so $(y-2)^2 = 99$, and because we're looking for a fourth-quadrant point, the answer is $y = 2 - \sqrt{99} = \mathbf{2 - 3\sqrt{11}}$.

Team Round

1. $\boxed{\frac{\mathbf{5}}{\mathbf{12}}}$ Factor $2020 = 2^2 \cdot 5 \cdot 101$, so there are $3 \cdot 2 \cdot 2 = 12$ positive integer factors of 2020. Of these, five are less than 20 (namely 1, 2, 4, 5, and 10). Thus, the desired probability is $\dfrac{\mathbf{5}}{\mathbf{12}}$.

2. $\boxed{\frac{\mathbf{75}}{\mathbf{286}}}$ For the median to be 7, four numbers must be less than 7 and four numbers must be greater than 7. This can be done in $\dbinom{6}{4} \cdot \dbinom{7}{4} = \dbinom{6}{2} \cdot \dbinom{7}{3} = \dfrac{6 \cdot 5}{2 \cdot 1} \cdot \dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 15 \cdot 35$ ways. The sample space has size $\dbinom{14}{9} = \dbinom{14}{5} = \dfrac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 14 \cdot 13 \cdot 11$, so the desired probability is $\dfrac{15 \cdot 35}{14 \cdot 13 \cdot 11} = \dfrac{\mathbf{75}}{\mathbf{286}}$.

3. $\boxed{\mathbf{321}}$ Proceed by cases. Suppose that all four letter are distinct. In this case, there are $\dbinom{10}{4} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 3 \cdot 7 = 210$ sets of four letters.
Next, suppose there is exactly one pair among the four letters. In this case, there are three possible pairs and $\dbinom{9}{2} = 36$ ways to choose the other two letters, so there are $3 \cdot 36 = 108$ sets of this type.
Now, suppose there are two pairs of letters. There are $\dbinom{3}{2} = 3$ ways to choose the pairs, and this uniquely determines the sets.
The final answer is $210 + 108 + 3 = \mathbf{321}$.

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