Greater Boston Mathematics League

G B M L

ANSWERS

1. ______________________

2. ______________________

3. ______________________

Meet 2 – November 13, 2019

The word "compute" calls for an exact answer in simplest form.

Round 1: Arithmetic – Open

1. Consider the infinite set of integers
$\{2 \cdot 3 \cdot 4 \cdot 5, 3 \cdot 4 \cdot 5 \cdot 6, 4 \cdot 5 \cdot 6 \cdot 7, \cdots, n \cdot (n+1) \cdot (n+2) \cdot (n+3), \cdots\}$. Compute the greatest common factor for this set of integers.

2. Let A, B, and C be three numbers such that $A = 2\sqrt{38}$, $B = \dfrac{\sqrt{1351}}{3}$, and $C = \sqrt{486} - \sqrt{96}$. Arrange the three numbers from least to greatest. Specify the ordered triple that lists the numbers from least to greatest. That is, if you believe that $A < B < C$, write as your answer (A, B, C).

3. Compute the least positive integer N such that $\sqrt{6! \cdot 7! \cdot N}$ is a perfect square.

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Round 2: Algebra – Simultaneous Linear Equations, Word Problems, Matrices

1. George's age is three times the age of his daughter Marah. In five years, the ratio of their ages will be $5:2$. Compute Marah's current age.

2. Compute the value of x for which $1234x + 923y = 301$ and $784x + 1095y = 1717$.

3. Let $A = \begin{bmatrix} 2 & 0 \\ 1 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 7 \\ 7 & 6 \end{bmatrix}$. Given that $B^{-1}A^{-1} = \frac{-1}{k}\begin{bmatrix} w & x \\ y & 2 \end{bmatrix}$, compute $k + w + x + y$.

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Round 3: Geometry – Angles and Triangles

1. The supplement of the complement of $\angle ABC$ measures 152°. Compute $m\angle ABC$ in degrees.

2. In $\triangle QUA$, D is the midpoint of $\overline{QA}$. Given that $QA = 18$, $DU = 9$, and $QU = 10$, compute AU.

3. Jimmy places n points P_0, P_1, P_2, $\cdots$, P_{n-1} on a circle in such a way that $m\angle P_0P_1P_2 = m\angle P_2P_3P_4 = \cdots = m\angle P_{n-2}P_{n-1}P_0 = 162^\circ$ and $m\angle P_1P_2P_3 = m\angle P_3P_4P_5 = \cdots = m\angle P_{n-1}P_0P_1 = 168^\circ$. Compute n.

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The word "compute" calls for an exact answer in simplest form.

Round 4: Algebra 2 – Quadratic Equations (including Theory)

1. For some integer b, one factor of $5x^2 + bx - 80$ is $x + 8$. Compute b.

2. Suppose that $i = \sqrt{-1}$. Given that $-3 + i$ is a root of $x^3 + cx - 60 = 0$ for some integer c, compute c.

3. A 30-inch piece of wire is cut into two pieces. One piece is bent to form a square and the other piece is bent to form an isosceles right triangle. Given that the area of the square is equal to the area of the isosceles right triangle, compute the length of a leg of the isosceles right triangle in the form $\dfrac{a\sqrt{2} - b}{c}$ where a, b, and c are positive integers.

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Round 5: Trigonometry – Trig Equations

1. Compute the number of values of x for which $\sin x = 0.2019$ on the interval $0 \le x \le 540°$.

2. Solve on $0 \le x < 360°$: $\sin^2 x + 3\cos^2 x = 1$.

3. Given that $1 + \cos^2\theta + \cos^4\theta + \cos^6\theta + \cdots + \cos^{2n}\theta + \cdots = 2019$, compute $\cos(2\theta)$.

Greater Boston Mathematics League

G B M L

ANSWERS

(3 pts) 1. ______________________________

(3 pts) 2. ______________________________

(4 pts) 3. ______________________________

Meet 2 – November 13, 2019

The word "compute" calls for an exact answer in simplest form.

Team Round

1. For a first quadrant angle x, $\sin 2x = \dfrac{5}{4}\cos x$. Compute $\cot x$.

2. Compute the sum of all four-digit natural numbers of the form $2\,\underline{X}\,\underline{Y}\,9$ that are divisible by 99.

3. Seven numbers, each 1 or -1, are listed in a row in such a way that the sum of the first k numbers in the list is never negative for any k with $1 \leq k \leq 7$. For example, 1 -1 1 1 -1 -1 1 is a valid sequence because the successive partial sums are 1, 0, 1, 2, 1, 0, 1, but 1 1 -1 -1 -1 1 1 is not valid because the fifth partial sum is -1. Compute the number of valid lists formed in this way.

GBML Answer Sheet

Meet #2

Round 1.

1. 24

2. (C, B, A)

3. 8575

Round 2.

1. 15

2. -2

3. 757

Round 3.

1. 62

2. $4\sqrt{14}$

3. 24

Round 4.

1. 30

2. -26

3. $\frac{45\sqrt{2}-30}{7}$

Round 5.

1. 4

2. $90, 270$ (need both)

3. $\frac{2017}{2019}$

Team Round

1. $\frac{\sqrt{39}}{5}$

2. 2079

3. 35

GBML Contest #2. SOLUTIONS

Round 1: Arithmetic – Open

1. $\boxed{\mathbf{24}}$ For the numbers n, $n+1$, and $n+2$, exactly one of them will be divisible by 3. Notice that for each group of four factors making the numbers in the given set, two of them are even, and one of those must be a multiple of 4, so $2 \cdot 4 = 8$ must be a factor of each of these numbers. No other prime number must appear in the factors of each of these numbers. The answer is $3 \cdot 8 = \mathbf{24}$.

2. $\boxed{(\mathbf{C}, \mathbf{B}, \mathbf{A})}$ Notice that $C = 9\sqrt{6} - 4\sqrt{6} = 5\sqrt{6}$. Now, square the three numbers. This will preserve the order. Compute $A^2 = 4 \cdot 38 = 152$, $B^2 = 1351/9 \approx 150.111$, and $C^2 = 25 \cdot 6 = 150$. The order is $C < B < A$, so the ordered triple is $(\mathbf{C}, \mathbf{B}, \mathbf{A})$.

3. $\boxed{\mathbf{8575}}$ Because $7! = 7 \cdot 6!$, $\sqrt{6! \cdot 7! \cdot N} = \sqrt{6! \cdot 6! \cdot 7 \cdot N}$ is a perfect square. This means that $6! \cdot \sqrt{7N}$ is a perfect square, so $720\sqrt{7N} = 144 \cdot 5 \cdot \sqrt{7N}$ is a perfect square. Because 144 is a perfect square, so is $5\sqrt{7N}$. Thus N has a factor of 25 and a factor of $7^3 = 343$, so N is at least $25 \cdot 343 = \mathbf{8575}$.

Round 2: Algebra – Simultaneous Linear Equations, Word Problems, Matrices

1. $\boxed{\mathbf{15}}$ Let Marah's age be M and George's age be $3M$. Then
$\frac{3M+5}{M+5} = \frac{5}{2} \rightarrow 6M + 10 = 5M + 25$, so $M = \mathbf{15}$.

2. $\boxed{\mathbf{-2}}$ Adding the equations, $2018x + 2018y = 2018$, so $x + y = 1 \rightarrow y = 1 - x$. Substituting, $1234x + 923(1-x) = 301$, which implies $311x = -622$, so $x = -\mathbf{2}$.

3. $\boxed{\mathbf{757}}$ Use the property that $B^{-1}A^{-1} = (AB)^{-1}$. The matrix AB is $\begin{bmatrix} 2 & 14 \\ 64 & 61 \end{bmatrix}$, so
$B^{-1}A^{-1} = (AB)^{-1} = \frac{1}{122 - 896} \begin{bmatrix} 61 & -14 \\ -64 & 2 \end{bmatrix}$. The desired inverse matrix is
$\frac{-1}{774} \begin{bmatrix} 61 & -14 \\ -64 & 2 \end{bmatrix}$, so the answer to the question is $774 + 61 - 14 - 64 = \mathbf{757}$.

Round 3: Geometry – Angles and Triangles

1. $\boxed{\mathbf{62}}$ Because the supplement of the complement of $\angle ABC$ measures $152°$, the complement of $\angle ABC$ measures $180 - 152 = 28°$. Thus $m\angle ABC = 90 - 28 = \mathbf{62}°$.

2. $\boxed{\mathbf{4\sqrt{14}}}$ Notice that $QD = AD = UD = 9$. Construct point R on $\overrightarrow{UD}$ such that $UR = 18$. Then $QUAR$ is a rectangle. Thus, $AU = \sqrt{18^2 - 10^2} = \sqrt{224} = \mathbf{4\sqrt{14}}$.

3. $\boxed{\mathbf{24}}$ Construct the n-gon $P_0P_1P_2 \cdots P_{n-1}$. Exterior angles at P_1, P_3, $\cdots$, and P_{n-1} measure $18°$. Exterior angles at P_0, P_2, $\cdots$, and P_{n-2} measure $12°$. The measures of all exterior angles of an n-gon add to $360°$, so solve $\frac{n}{2}(18) + \frac{n}{2}(12) = 360$ to obtain $n = \mathbf{24}$.

Round 4: Algebra 2 – Quadratic Equations (including Theory)

1. $\boxed{\mathbf{30}}$ Because $x+8$ is a factor, -8 is a root of $5x^2+bx-80=0$. Substituting, $5(-8)^2-8b-80=0$, which implies $240-8b=0 \to b=\mathbf{30}$.

2. $\boxed{\mathbf{-26}}$ Because the coefficients of the polynomial are all integers, the complex roots occur in conjugate pairs. Thus, $-3-i$ is a root also. The product of the two known roots is $(-3+i)(-3-i)=9-i^2=10$. The product of the three roots of the cubic is $-(-60)=60$, so $\frac{60}{10}=6$ is a root. Substituting $x=6$ yields $216+6c-60=0 \to 6c=-156 \to \mathbf{c=-26}$.

Alternate Solution: Because $(-3+i)$ is a root of $x^3+cx-60=0$, substitute to obtain $(-3+i)^3+c(-3+i)-60=0 \to (-3+i)^2(-3+i)+c(-3+i)-60=0$, which is equivalent to $(8-6i)(-3+i)+c(-3+i)-60=0$, which implies $-24+26i+6-60+c(-3+i)=0$. Divide to obtain $c=-\left(\dfrac{-78+26i}{-3+i}\right)=\dfrac{-26(-3+i)}{-3+i}=-26$.

3. $\boxed{\dfrac{\mathbf{45\sqrt{2}-30}}{\mathbf{7}}}$ Let one leg of the right isosceles triangle have length x; then the sides of the right isosceles triangle are x, x, and $x\sqrt{2}$. This implies that the sides of the square each measure $\dfrac{30-2x-x\sqrt{2}}{4}$. The problem implies that $\left(\dfrac{30-2x-x\sqrt{2}}{4}\right)^2=\dfrac{1}{2}x^2$, or $(30-2x-x\sqrt{2})^2=8x^2$. Expand the brackets and rearrange to obtain $(4\sqrt{2}-2)x^2+(-60\sqrt{2}-120)x+900=0$. Now use the quadratic formula to obtain $x=\dfrac{60\sqrt{2}+120\pm\sqrt{(60\sqrt{2}+120)^2-4(4\sqrt{2}-2)(900)}}{8\sqrt{2}-4}$ or $x=\dfrac{60\sqrt{2}+120\pm\sqrt{28800}}{8\sqrt{2}-4}=\dfrac{60\sqrt{2}+120\pm120\sqrt{2}}{8\sqrt{2}-4}$. Recognize that $\dfrac{60\sqrt{2}+120+120\sqrt{2}}{8\sqrt{2}-4}$ is greater than 30, so it is rejected, and the value of x is $x=\dfrac{120-60\sqrt{2}}{8\sqrt{2}-4}=\dfrac{\mathbf{45\sqrt{2}-30}}{\mathbf{7}}$.

Alternate Solution: Let the square have side length x and the isosceles right triangle have leg length y. Because the areas are equal, $x^2=\dfrac{1}{2}y^2$ and $x=\dfrac{y}{\sqrt{2}}$. Then, $4x+y(2+\sqrt{2})=30 \leftrightarrow y(3\sqrt{2}+2)=30$. This means that $y=\dfrac{30}{3\sqrt{2}+2}=\dfrac{30(3\sqrt{2}-2)}{18-4}=\dfrac{45\sqrt{2}-30}{7}$.

Round 5: Trigonometry: Trig Equations

1. $\boxed{4}$ The equation will have a solution in Quadrant I and a solution in Quadrant II. Those quadrants are "repeated" in the interval $360° \leq x \leq 540°$, so the answer is **4**.

2. $\boxed{\mathbf{90}, \mathbf{270} \text{ (need both)}}$ Rewrite the given equation as $\sin^2 x + \cos^2 x + 2\cos^2 x - 1 = 0 \rightarrow \cos(2x) = -1$. Therefore, $2x = 180$ or $2x = 540$, so $x \in \{\mathbf{90}, \mathbf{270}\}$.

3. $\boxed{\dfrac{\mathbf{2017}}{\mathbf{2019}}}$ The left-hand side of the given equation is a geometric series with first term 1 and common ratio $\cos^2\theta$, so the sum of the left hand side is $\dfrac{1}{1-(\cos^2\theta)} = \dfrac{1}{\sin^2\theta} = 2019 \rightarrow \sin^2\theta = \dfrac{1}{2019}$. Thus $\cos(2\theta) = 1 - 2\sin^2\theta = 1 - \dfrac{2}{2019} = \dfrac{\mathbf{2017}}{\mathbf{2019}}$.

Team Round

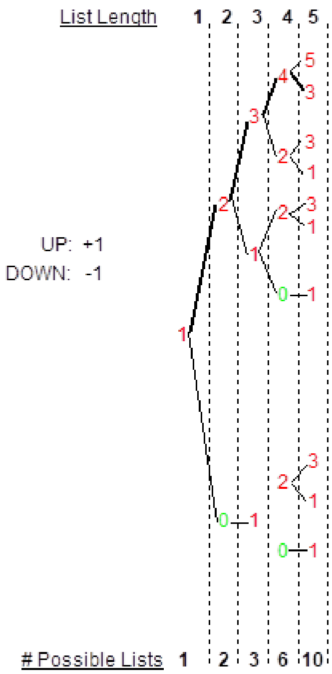
1. $\boxed{\dfrac{\sqrt{\mathbf{39}}}{\mathbf{5}}}$ Substituting from the double-angle formula, $2\sin x\cos x = \dfrac{5}{4}\cos x$. Because x is in the first quadrant, $sinx > 0$, and so this equation is equivalent to $\sin x = \dfrac{5}{8}$. This implies that $\cos^2 x + \left(\dfrac{5}{8}\right)^2 = 1 \rightarrow \cos x = \dfrac{\sqrt{39}}{8}$. Using the fact that $\cot x = \dfrac{\cos x}{\sin x}$, the answer is $\dfrac{\sqrt{\mathbf{39}}}{\mathbf{5}}$.

2. $\boxed{\mathbf{2079}}$ If a number is divisible by 99, it is divisible by 9 and by 11. If the number is divisible by 9, then $11 + X + Y$ is divisible by 9, so $X + Y \in \{7, 16\}$. If the number is divisible by 11, then $2 - X + Y - 9 = Y - X - 7$ is divisible by 11, and so $Y - X = -4$ or $Y - X = 7$. Pairing each of the first conditions with each of the second conditions makes for four possible scenarios:
If $X + Y = 7$ and $Y - X = -4$, then $2Y = 3$ which is impossible.
If $X + Y = 16$ and $Y - X = -4$, then $2Y = 12$ and $Y = 6$ and $X = 10$ which is impossible.
If $X + Y = 7$ and $Y - X = 7$, then $2Y = 14$ and $Y = 7$ and $X = 0$, resulting in 2079.
If $X + Y = 16$ and $Y - X = 7$, then $2Y = 23$ which is impossible.
Thus, there is only one number meeting the requirements, and the answer is **2079**.

3. $\boxed{\mathbf{35}}$ The first digit must be 1 for the sequence to be valid, and there are 64 sequences of this type. It is impossible for the first partial sum of -1 to occur after an even number of terms. Therefore, consider the ways that the first partial sum can be -1. There are 16 sequences for which the first partial sum of -1 occurs after three terms; these are sequences of the form 1 -1 -1 A B C D. There are 8 sequences for which the first partial sum of -1 occurs after five terms; 4 of these are sequences of the form 1 -1 1 -1 -1 E F and 4 of these are sequences of the form 1 1 -1 -1 -1 G H. There are 5 sequences for which the first partial sum of -1 does not occur until the final term: 1 1 1 -1 -1 -1 -1, 1 1 -1 1 -1 -1 -1, 1 1 -1 -1 1 -1 -1, 1 -1 1 1 -1 -1 -1, and 1 -1 1 -1 1 -1 -1. There are therefore $64 - 16 - 8 - 5 = \mathbf{35}$ valid sequences.

Alternate Solution: The tree diagram below highlights the basic principle. When moving to the right (adding another number to the list), there are two possible new partial sums, unless the partial sum at the current level is 0. Moving to the right and up adds 1 to the list, and moving to the right and down adds -1 to the list. For example, the list represented by the bolded path is $\{1, 1, 1, 1, -1\}$ corresponding to a partial sum of 3 for a list of 5 numbers.

There are 6 possible lists with 4 numbers. This gives 10 possible lists with 5 numbers, all of which have nonzero partial sums, implying that there are 20 lists with 6 numbers. Of these, 5 have partial sums of 0 and 15 have positive partial sums. Thus, there will be $15 \cdot 2 + 5 = 35$ lists with 7 numbers.



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