Greater Boston Mathematics League

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

Meet 1 – October 9, 2019

The word "compute" calls for an exact answer in simplest form.

Round 1: Arithmetic – Open

1. Let the set $S = \{3, 5, 7, 11, 13, 16, 17, 23, 25, 29\}$. Compute the least number of distinct numbers in S that add to 50.

2. When written as a decimal, $\dfrac{20}{19} = 1.\overline{052631578947368421}$. Compute the sum of the first 2019 digits after the decimal point in the decimal expansion of $\dfrac{20}{19}$.

3. Given that $2019_{N+3} = 15606_{N-3}$, compute N^2, and write your answer in base $N + 3$. *If necessary, use A to represent the digit ten, B to represent the digit eleven, and so on.*

Greater Boston Mathematics League

GBML

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

Meet 1 – October 9, 2019

The word "compute" calls for an exact answer in simplest form.

Round 2: Algebra 1 – Problem Solving (Word Problems)

1. At StuffMart, Terry can buy 3 pairs of pants and 7 pairs of socks for \$75. Terry can also buy 4 pairs of pants and 1 pair of socks for \$75. Compute the number of dollars Terry would pay for 1 pair of pants and 1 pair of socks.

2. Jenna and Jimmy always run their training runs along the same course, starting at the same point and ending at the same point. Jimmy runs 2 mph faster than Jenna. If Jimmy starts 30 minutes after Jenna, they will both reach the finish line at the same time, 2 hours after Jimmy starts. Compute the length of the training run in miles.

3. Three real numbers multiply to 16848. The second number is three times the first. The third number is three more than the second number. Compute the sum of the three numbers.

Greater Boston Mathematics League

G B M L

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

Meet 1 – October 9, 2019

The word "compute" calls for an exact answer in simplest form.

Round 3: Algebra 1 – Exponents and Radicals

1. Compute the value of $\sqrt[4]{49} \cdot \sqrt[6]{343}$.

2. Compute the value of $\left(2019^{2020} + 2019^{-2020}\right)^2 - \left(2019^{2020} - 2019^{-2020}\right)^2$.

3. Compute the value of $\sqrt{33 - 20\sqrt{2}} + \sqrt{17 + 12\sqrt{2}}$.

Greater Boston Mathematics League

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

Meet 1 – October 9, 2019

The word "compute" calls for an exact answer in simplest form.

Round 4: Algebra 2 – Factoring

1. Given that $x^2 + 2x + 4 = 0$, compute the value of x^6.

2. Factor as the product of two quadratic trinomials each of whose leading coefficient is 1: $x^4 + x^2 + 1$.

3. Factor as the product of two trinomials each of whose leading coefficients is 1: $x^3 + ax^2 + 7x^2 + 3ax + 13x + a + 4$.

Greater Boston Mathematics League

G B M L

ANSWERS

1. ______________________

2. ______________________

3. ______________________

Meet 1 – October 9, 2019

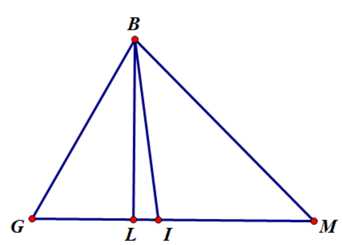
The word "compute" calls for an exact answer in simplest form.

Round 5: Trigonometry – Angular and Linear Velocity / Right Triangle Trigonometry

1. In right triangle BUG, $\cos G = \dfrac{2}{5}$. Compute $\sin G + \tan G$.

2. A car is traveling at 5π meters per second. The wheel radius is 330 millimeters. If the car travels for 22 minutes, compute the number of revolutions the wheel makes in that time.

3. In the diagram of $\triangle GBM$, $\overline{BL} \perp \overline{GM}$ and $\overline{BI}$ is an angle bisector for $\angle GBM$. Given that $m\angle BGM = 60°$ and $m\angle BMG = 45°$, compute $\cot(\angle BIG)$.



Greater Boston Mathematics League

G B M L

ANSWERS

(3 pts) 1. ______________________________

(3 pts) 2. ______________________________

(4 pts) 3. ______________________________

Meet 1 – October 9, 2019

The word "compute" calls for an exact answer in simplest form.

Team Round

1. Given that $\underline{1}\,\underline{2}\,\underline{2}\,\underline{A}\,\underline{7}\,\underline{7} \cdot 2646 = \underline{3}\,\underline{2}\,\underline{3}\,\underline{2}\,\underline{B}\,\underline{0}\,\underline{3}\,\underline{4}\,\underline{2}$, compute $A + B$.

2. Compute the greatest root of $x^4 - x^3 - 25x^2 + 9x + 144 = 0$.

3. The side lengths of $\triangle BOS$ are whole numbers and $\angle O$ is a right angle. Given that $\csc S = w + \dfrac{1}{16}$ where w is a whole number, compute $\sec S$.

GBML Answer Sheet

Meet #1

Round 1.

1. 3

2. 9079

3. 79

Round 2.

1. 21

2. 20

3. 87

Round 3.

1. 7

2. 4

3. 8

Round 4.

1. 64

2. $(x^2 - x + 1)(x^2 + x + 1)$

3. $(x^2 + 3x + 1)(x + 4 + a)$

Round 5.

1. $\dfrac{7\sqrt{21}}{10}$

2. 10000

3. $\sqrt{2} - 2 + \sqrt{6} - \sqrt{3}$

Team Round

1. 9

2. $\dfrac{1 + \sqrt{65}}{2}$

3. $\dfrac{65}{63}$ or $1\dfrac{2}{63}$ or $1.\overline{031746}$

GBML Contest #1. SOLUTIONS

Round 1: Arithmetic – Open

1. $\boxed{3}$ A quick check determines that neither one nor two distinct numbers will add to 50. Try three numbers; note that $29 + 16 + 5 = 50$. Therefore, the answer is **3**.

2. $\boxed{9079}$ The repetend 052631578947368421 has sum 81 and 18 digits. Because $2019 = 112 \cdot 18 + 3$, the repetend appears 112 times after the initial 1, and that makes 2016 digits. The desired sum is $112 \cdot 81 + 0 + 5 + 2 = \mathbf{9079}$.

3. $\boxed{79}$ The given equation implies $2(N+3)^3 + (N+3) + 9 = (N-3)^4 + 5(N-3)^3 + 6(N-3)^2 + 6$, or $2N^3 + 18N^2 + 55N + 66 = N^4 - 7N^3 + 15N^2 - 9N + 6$, or $N^4 - 9N^3 - 3N^2 - 64N - 60 = 0$. By Vieta's formulas, the product of the roots is 60, and because N is an integer, N is a factor of 60. Also, $N \geq 7$ because 9 is a digit in base $N + 3$. By inspection, $N = 10$ is a root. By algebra, $N^4 - 9N^3 - 3N^2 - 64N - 60 = (N-10)(N^3 + N^2 + 7N + 6) = 0$ The latter cubic cannot have any roots in the positive integers and so no other values of N are possible. The answer is 100 expressed in base 13. Because $100 = 13 \cdot 7 + 9$, the answer is **79**.

Round 2: Algebra 1 – Problem Solving (Word Problems)

1. $\boxed{21}$ Suppose a pair of pants costs \$$P$ and a pair of socks costs \$$S$. Then $3P + 7S = 4P + 1S \to 6S = P$. Substituting, $18S + 7S = 75 \to S = 3$ and $P = 6 \cdot 3 = 18$. The cost of a pair of pants and a pair of socks in dollars is $3 + 18 = \mathbf{21}$.

2. $\boxed{20}$ Let x be the length of the run. Then, using the fact that speed is the quotient of distance and time, and using the fact that Jimmy's speed is 2 mph faster than Jenna's, the problem situation implies $\frac{x}{2.5} + 2 = \frac{x}{2}$. This implies $2x + 10 = 2.5x \to 10 = 0.5x \to x = \mathbf{20}$ miles.

3. $\boxed{87}$ Call the three numbers x, $3x$, $3x + 3$, so the product is $9x^2(x+1) = 16848$, which implies $x^2(x+1) = 1872$. Because $1872 = 2^4 \cdot 3^2 \cdot 13 = 12^2 \cdot 13$, $x = 12$ and the sum of the three numbers is $12 + 36 + 39 = \mathbf{87}$.

Round 3: Algebra 1 – Exponents and Radicals

1. $\boxed{7}$ The given expression is equivalent to $\sqrt[4]{7^2} \cdot \sqrt[6]{7^3} = 7^{2/4} \cdot 7^{3/6} = 7^{1/2+1/2}$. This is $7^1 = \mathbf{7}$.

2. $\boxed{4}$ Let $A = 2019^{2020} + 2019^{-2020}$ and $B = 2019^{2020} - 2019^{-2020}$. The given expression is $A^2 - B^2 = (A+B)(A-B)$, or $(2 \cdot 2019^{2020}) \cdot (2 \cdot 2019^{-2020})$, which is **4**.

3. $\boxed{8}$ First, $\sqrt{33 - 20\sqrt{2}}$ can be expressed as $a + b\sqrt{2}$, and squaring this expression implies $a^2 + 2b^2 = 33$ and $2ab = -20$. Inspection plus some trial and error obtains $a = 5$ and $b = -2$. Thus $\sqrt{33 - 20\sqrt{2}} = 5 - 2\sqrt{2}$.
Now, in a similar way, $\sqrt{17 + 12\sqrt{2}}$ can be expressed as $c + d\sqrt{2}$, and squaring this expression implies $c^2 + 2d^2 = 17$ and $2cd = 12$. Inspection plus some trial and error obtains $c = 3$ and $d = 2$. Thus $\sqrt{17 + 12\sqrt{2}} = 3 + 2\sqrt{2}$.
Adding obtains a sum of $5 - 2\sqrt{2} + 3 + 2\sqrt{2} = \mathbf{8}$.

Round 4: Algebra 2 – Factoring

1. $\boxed{\mathbf{64}}$ Notice that $(x-2)(x^2+2x+4) = x^3-8 = 0$, so $x^6 = (x^3)^2 = 8^2 = \mathbf{64}$. Because the complex roots of the given quadratic equation are two of the cube roots of 8, raising either of these values to the sixth power results in an answer of 64.

2. $\boxed{\boldsymbol{(x^2-x+1)(x^2+x+1)}}$ Notice that the given expression is equivalent to $x^4+2x^2+1-x^2 = (x^2+1)^2-x^2$. Now, factor this as the difference of two squares to obtain $\boldsymbol{(x^2-x+1)(x^2+x+1)}$.

3. $\boxed{\boldsymbol{(x^2+3x+1)(x+4+a)}}$ First, group the terms with a factor of $\boldsymbol{a}$ together to obtain $\boldsymbol{x^3+7x^2+13x+4+a(x^2+3x+1)}$. A good guess is that the cubic has $\boldsymbol{x+4}$ as one of its factors, and indeed that is the case, so this is equivalent to $\boldsymbol{(x^2+3x+1)(x+4)+a(x^2+3x+1)}$, or $\boldsymbol{(x^2+3x+1)(x+4+a)}$.

Alternate Solution: Set the expression equal to zero. Regrouping, the equation can be written as $\boldsymbol{x^3+(a+7)x^2+(3a+13)x+(a+4)=0}$. Applying the Rational Root Theorem, the only possible rational roots are $\boldsymbol{\pm 1}$ and $\boldsymbol{\pm(a+4)}$. By synthetic substitution, $\boldsymbol{-(a+4)}$ is a root, so $\boldsymbol{(x+a+4)}$ is a factor, and the other trinomial factor is $\boldsymbol{x^2+3x+1}$, which results from the synthetic division.

$$
\begin{array}{r|rrrr}
 & 1 & a+7 & 3a+13 & a+4 \\
 & & -a-4 & -3a-12 & -a-4 \\
\hline
-a-4 & 1 & 3 & 1 & 0
\end{array}
$$

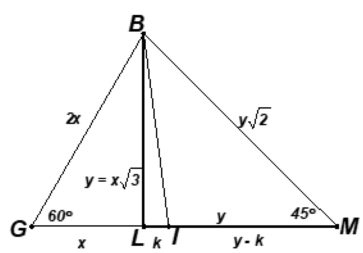
Round 5: Trigonometry – Angular and Linear Velocity / Right Triangle Trigonometry

1. $\boxed{\dfrac{\mathbf{7\sqrt{21}}}{\mathbf{10}}}$ The "opposite" side measures $\sqrt{5^2-2^2}=\sqrt{21}$. Thus, $\sin G+\tan G=\dfrac{\sqrt{21}}{5}+\dfrac{\sqrt{21}}{2}=\dfrac{\mathbf{7\sqrt{21}}}{\mathbf{10}}$.

2. $\boxed{\mathbf{10000}}$ The circumference of the wheel is 660π millimeters. The car is traveling at 5000π millimeters per second or $300,000\pi$ millimeters per minute. Because the car travels for 22 minutes, the wheel makes $\dfrac{300000\pi\cdot 22}{660\pi}=\mathbf{10000}$ revolutions.

3. $\boxed{\sqrt{\mathbf{2}}-\mathbf{2}+\sqrt{\mathbf{6}}-\sqrt{\mathbf{3}}}$ Suppose $BL=\sqrt{3}$. Then $BG=2$ and $BM=\sqrt{3}\cdot\sqrt{2}=\sqrt{6}$. Also, $GM=GL+LM=1+\sqrt{3}$. By the Angle Bisector Theorem, $GI:IM=2:\sqrt{6}$, so solve $2x+\sqrt{6}x=1+\sqrt{3}$ to obtain $GI=\sqrt{6}-2\sqrt{3}+3\sqrt{2}-2$. Thus $LI=GI-GL=\sqrt{6}-2\sqrt{3}+3\sqrt{2}-2-1=\sqrt{6}-2\sqrt{3}+3\sqrt{2}-3$, and $\cot(\angle BIG)=\dfrac{LI}{BL}=\dfrac{\sqrt{6}-2\sqrt{3}+3\sqrt{2}-3}{\sqrt{3}}=\dfrac{3\sqrt{2}-6+3\sqrt{6}-3\sqrt{3}}{3}$, or $\sqrt{\mathbf{2}}-\mathbf{2}+\sqrt{\mathbf{6}}-\sqrt{\mathbf{3}}$.

Alternate Solution: By the Angle Bisector Theorem, $\dfrac{x+k}{2x}=\dfrac{y-k}{y\sqrt{2}}=\dfrac{x\sqrt{3}-k}{x\sqrt{6}}$. This is equivalent to $x\sqrt{6}+k\sqrt{6}=2x\sqrt{3}-2k\leftrightarrow x(2\sqrt{3}-\sqrt{6})=k(\sqrt{6}+2)$ so $\dfrac{k}{x}=\dfrac{2\sqrt{3}-\sqrt{6}}{\sqrt{6}+2}$. Now, note that $\cot(\angle BIG)=\dfrac{k}{x\sqrt{3}}=\dfrac{2\sqrt{3}-\sqrt{6}}{(\sqrt{6}+2)\sqrt{3}}=\dfrac{2-\sqrt{2}}{\sqrt{6}+2}\cdot\dfrac{\sqrt{6}-2}{\sqrt{6}-2}$ or $\dfrac{2\sqrt{6}-4-2\sqrt{3}+2\sqrt{2}}{2}=\sqrt{2}-2+\sqrt{6}-\sqrt{3}$.



Team Round

1. $\boxed{\mathbf{9}}$ Notice that 2646 is a multiple of 9, so $\underline{3}\,\underline{2}\,\underline{3}\,\underline{2}\,\underline{B}\,\underline{0}\,\underline{3}\,\underline{4}\,\underline{2}$ is a multiple of 9. The digit sum of $\underline{3}\,\underline{2}\,\underline{3}\,\underline{2}\,\underline{B}\,\underline{0}\,\underline{3}\,\underline{4}\,\underline{2}$ is $19+B$, and the only digit that makes $19+B$ a multiple of 9 is $B=8$. Now, notice that $\underline{3}\,\underline{2}\,\underline{3}\,\underline{2}\,\underline{8}\,\underline{0}\,\underline{3}\,\underline{4}\,\underline{2}$ is a multiple of 11 because the "alternating digit sum" $3-2+3-2+8-0+3-4+2=11$ is a multiple of 11. Also notice that 2646 is not a multiple of 11, so $\underline{1}\,\underline{2}\,\underline{2}\,\underline{A}\,\underline{7}\,\underline{7}$ must be a multiple of 11. The alternating digit sum of $\underline{1}\,\underline{2}\,\underline{2}\,\underline{A}\,\underline{7}\,\underline{7}$ is $1-2+2-A+7-7=1-A$, and the only digit that makes $1-A$ a multiple of 11 is $A=1$. The sum $A+B$ is $1+8=\mathbf{9}$.

2. $\boxed{\dfrac{\mathbf{1}+\sqrt{\mathbf{65}}}{\mathbf{2}}}$ Factor by rewriting the equation as $x^4-25x^2+144+(-x^3+9x)=(x^2-9)(x^2-16)-x(x^2-9)=0$. Removing the greatest common factor, this is equivalent to $(x^2-9)(x^2-x-16)=0$. The first factor yields roots of ± 3, and the second yields roots of $\dfrac{1\pm\sqrt{65}}{2}$. Of these, the greatest is $\dfrac{\mathbf{1}+\sqrt{\mathbf{65}}}{\mathbf{2}}$.

3. $\boxed{\dfrac{\mathbf{65}}{\mathbf{63}}}$ Because $\csc S=\dfrac{16w+1}{16}$, the leg $\overline{BO}$ measures 16 (without loss of generality). Call the lengths $SB=c$ and $OS=a$ for ease. Then $c^2-a^2=16^2=256$. This implies $(c-a)(c+a)=256$. Consider systems of equations $c-a=k$ and $c+a=\dfrac{256}{k}$ where k is an even factor of 256; note that if k is odd, there is no solution in the whole numbers as required by the problem statement. Every system results in c being even except for $c-a=2$ and $c+a=128$, which solves to obtain $c=65$ and $a=63$ and $\sec S=\dfrac{\mathbf{65}}{\mathbf{63}}$.

Author: George Reuter - coachreu@gmail.com

Reviewers: Mike Schockett - mschockett@comcast.net and Dick Olson - olson.re@gmail.com