Greater Boston Mathematics League

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ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

Meet 1 – October 14, 2020

The word "compute" calls for an exact answer in simplest form.

Calculators are not permitted in this round.

Round 1: Arithmetic – Open

1. The number of students in the GBML Fan Club went up by 25% on Monday, and then by 20% on Tuesday, and then by 50% on Wednesday. Given that there were 4545 students in the fan club at the end of Wednesday, compute the number of students in the fan club at the beginning of Monday.

2. There are two two-digit base-8 numbers $\underline{A}\,\underline{B}_8$ that are both divisible by 4 and 7. Compute the base-8 sum of these two numbers.

3. The number $2020^{2020} + 2021^{2021} + 2020 + 2021$ is written as a decimal number. Compute the remainder when this decimal number is divided by 100.

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Round 2: Algebra 1 – Problem Solving (Word Problems)

1. A change purse contains only dimes and quarters worth a total of \$3.80. Given that there are 20 coins in the purse, compute the number of quarters in the purse.

2. Brady's age is B years old. Brady has three children, and their ages add to B years old. Also, N years ago, Brady's age was three times the sum of his children's ages then. Compute $\frac{B}{N}$.

3. PJ is thinking of four positive integers a, b, c, and d, with $a < b < c < d$. His friend Ari asks for the four numbers. PJ says, "I won't tell you the numbers, but I will give you their six pairwise sums: 18, 15, 13, 12, 10, and 7. From this information, Ari was able to compute the four numbers. Compute $a^2 + b^2 + c^2 + d^2$.

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Round 3: Algebra 1 – Exponents and Radicals

1. Let $N = A^B + C^D$ be a **positive integer** where A, B, C, and D are distinct integers chosen from the set $\{-1, 2, -3, 4\}$. Compute the **minimum** possible value of N.

2. For positive values of x, compute $\dfrac{2^{x+1} \cdot 3^{x+2} \cdot 6^{x+3}}{9^{x+1} \cdot 4^{x-1}}$.

3. Compute $\sqrt{\dfrac{(\sqrt{3}+\sqrt{15})^2}{2} \cdot \left(1 - \dfrac{\sqrt{5}}{3}\right)}$.

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Round 4: Algebra 2 – Factoring

1. Factor $9x^2 + 4y^2 - 9z^2 - 12xy$ into the product of two trinomials with integer coefficients. For each polynomial, let the coefficient of x be positive.

2. Express $a^2 - ac - 4b^2 - 2bc$ as the product of two polynomials with integer coefficients. For each polynomial, let the coefficient of a be 1.

3. Compute $\dfrac{2 \cdot 2020^2 - 673 \cdot 2020 - 675}{4080399}$ as a fraction in simplest form.

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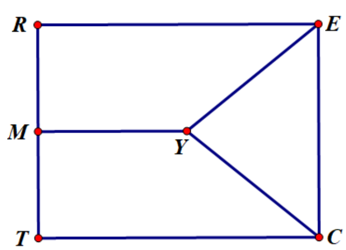
Round 5: Trigonometry – Angular and Linear Velocity / Right Triangle Trigonometry

1. Sam is driving at 46 feet per second. The diameter of each of Sam's wheels is 23 inches. Compute, to the <u>nearest whole number</u>, the number of minutes it takes a wheel to turn through 4800 revolutions.

2. Suppose that $\triangle GBM$ is drawn with $m\angle G = 30°$, $m\angle B = 105°$, and $BM = 6\sqrt{2}$. Compute the area of $\triangle GBM$.

3. Suppose that $RECT$ is a rectangle with $RE = 8$ and $RT = 6$ as shown in the diagram.



Let point M be on $\overline{RT}$ and point Y be in the interior of $RECT$, such that $YM = YE = YC$. Compute $\sin(\angle YEC)$.

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Meet 1 – October 14, 2020

The word "compute" calls for an exact answer in simplest form.

Calculators are not permitted in this round.

Team Round

1. Patrick is writing down every 4-digit multiple of 4 that can be made from the digits in the set $\{6, 4, 1, 2, 5\}$. Compute the number of numbers on Patrick's list.

2. Suppose that a, b, and c are positive integers such that $a(b + c) = 168$, $b(a + c) = 220$, and $c(a + b) = 234$. Compute $a + b + c$.

3. There is one value of x for which $\cos(\sin^{-1}(\cos(\tan^{-1} x))) = \frac{x}{2}$. Compute x. *Recall that the inverse sine and inverse tangent functions are one-to-one.*

GBML Answer Sheet

Meet #1

Round 1.

1. 2020

2. 124

3. 62

Round 2.

1. 12

2. 4

3. 193

Round 3.

1. 10

2. 1728

3. 2

Round 4.

1. $(3x-2y+3z)(3x-2y-3z)$

2. $(a-2b-c)(a+2b)$

3. $\frac{5}{3}$

Round 5.

1. 10

2. $18+18\sqrt{3}$

3. $\frac{55}{73}$

Team Round

1. 36

2. 31

3. $\sqrt{3}$

GBML Contest #1. SOLUTIONS

Round 1: Arithmetic – Open

1. $\boxed{\textbf{2020}}$ The number of students in the fan club is multiplied by $\frac{5}{4}$, and then by $\frac{6}{5}$, and then by $\frac{3}{2}$, so the multiplication factor is $\frac{5}{4} \cdot \frac{6}{5} \cdot \frac{3}{2} = \frac{9}{4}$. Solve $\frac{9N}{4} = 4545$ to obtain $N = \mathbf{2020}$.

2. $\boxed{\textbf{124}}$ If a number is divisible by 4 and 7, it is divisible by 28. The two numbers (in base 10) are 28 and 56, whose base-10 sum is 84. Converting to base 8, the sum of these numbers is $84 = 1 \cdot 64 + 2 \cdot 8 + 4$, so the answer is **124**.

3. $\boxed{\textbf{62}}$ Note that 2020^n ends in 00 for every n greater than 1. The rightmost digits of powers of 2021 cycle: 21, 41, 61, 81, 01, and so on. Because $2021 \equiv 1 \pmod 5$, it follows that 2021^{2021} ends in 21. Thus, the rightmost two digits of the given decimal number are $00 + 21 + 20 + 21 = \mathbf{62}$.

Round 2: Algebra 1 – Problem Solving (Word Problems)

1. $\boxed{\textbf{12}}$ If there were an equal number of dimes and quarters, there would be $10 \cdot (\$0.25 + \$0.10) = \$3.50$ in the purse. Each swap of a quarter for a dime raises the overall value of the coins by $\$0.25 - \$0.10 = \$0.15$. To reach the desired \$3.80, swap out two dimes for two quarters, so the answer is $10 + 2 = \mathbf{12}$.

2. $\boxed{\textbf{4}}$ Notice that N years ago, Brady's age was $B - N$ and his children's ages added to $B - 3N$. Therefore, $B - N = 3(B - 3N) \rightarrow B - N = 3B - 9N \rightarrow B = 4N$, so $\dfrac{B}{N} = \mathbf{4}$. This could happen if Brady was 40 with children of ages 15, 14, and 11; then, 10 years ago, Brady was 30 and his children were 5, 4, and 1.

3. $\boxed{\textbf{193}}$ The least pairwise sum is $a + b = 7$ and the greatest pairwise sum is $c + d = 18$. It is true either that $b + c = 15$ and $d + a = 10$ or $b + d = 15$ and $a + c = 10$.
Assume that $b + c = 15$ and $d + a = 10$. Then $(b + c) - (a + b) = 15 - 7 \rightarrow c - a = 8$. Because $b + c = 15$, it follows that $a + c = 12$, and thus $(c - a) + (a + c) = 8 + 12 \rightarrow c = 10$. This implies $a = 2$, $b = 5$, and $d = 8$. These integers do not satisfy $a < b < c < d$.
Assume that $b + d = 15$ and $a + c = 10$. Then $(b + d) - (a + b) = 15 - 7 \rightarrow d - a = 8$. It is not possible that $a + d = 13$ because that would lead to a non-integer value of d, so $a + d = 12$ and $2d = 20 \rightarrow d = 10$. This implies $a = 2$, $b = 5$, and $c = 8$. The answer is $2^2 + 5^2 + 8^2 + 10^2 = 4 + 25 + 64 + 100 = \mathbf{193}$.

Round 3: Algebra 1 – Exponents and Radicals

1. $\boxed{\textbf{10}}$ Since N must be an integer, neither -1 nor -3 may be an exponent. Thus, consider $(-1)^2 + (-3)^4$ and $(-1)^4 + (-3)^2$. Of these, the minimum is $(-1)^4 + (-3)^2 = \mathbf{10}$.

2. $\boxed{\textbf{1728}}$ The given expression is equivalent to $\dfrac{2^{x+1} \cdot 3^{x+2} \cdot 2^{x+3} \cdot 3^{x+3}}{3^{2x+2} \cdot 2^{2x-2}}$, which simplifies to $2^6 \cdot 3^3 = 4^3 \cdot 3^3 = 12^3 = \mathbf{1728}$.

3. $\boxed{\textbf{2}}$ Expand the numerator of the first fraction and simplify to obtain $\sqrt{\dfrac{6(3+\sqrt{5})}{2} \cdot \dfrac{3-\sqrt{5}}{3}}$, which is $\sqrt{(3+\sqrt{5})(3-\sqrt{5})} = \sqrt{9-5} = \mathbf{2}$.

Round 4: Algebra 2 – Factoring

1. $\boxed{\boldsymbol{(3x-2y+3z)(3x-2y-3z)}}$ The given expression is equivalent to $9x^2-12xy+4y^2-9z^2$, which can be rewritten as $(3x-2y)^2-9z^2$. This can be factored as the difference of two squares to obtain $(\mathbf{3x-2y+3z})(\mathbf{3x-2y-3z})$.

2. $\boxed{\boldsymbol{(a-2b-c)(a+2b)}}$ Note that $a^2-ac-4b^2-2bc$ can be rewritten as $a^2-2ab-ac+2ab-4b^2-2bc$, which equals $a(a-2b-c)+2b(a-2b-c)$, so the answer is $(\mathbf{a-2b-c})(\mathbf{a+2b})$.

3. $\boxed{\boldsymbol{\frac{5}{3}}}$ Let $x=2020$. Then the given expression is equivalent to $\frac{2x^2-673x-675}{x^2-1}$, which is equivalent to $\frac{(2x-675)(x+1)}{(x-1)(x+1)}=\frac{2x-675}{x-1}$. Substituting $x=2020$, this is equal to $\frac{3365}{2019}=\frac{5\cdot 673}{3\cdot 673}$, so the answer is $\boldsymbol{\frac{5}{3}}$.

Round 5: Trigonometry – Angular and Linear Velocity / Right Triangle Trigonometry

1. $\boxed{\mathbf{10}}$ The circumference of a wheel is 23π inches or $\frac{23\pi}{12}$ feet. To pass through 4800 revolutions would require the car to travel $400\cdot 23\pi=9200\pi$ feet. At 46 feet per second, this takes $9200\pi\div 46=200\pi$ seconds, or $\frac{10\pi}{3}$ minutes. This is approximately 10.47 minutes, so the answer is **10**.

2. $\boxed{\mathbf{18+18\sqrt{3}}}$ Drop the altitude from B to $\overline{GM}$. Notice that $\triangle BLM$ is a $45-45-90$ right triangle, so $BL=LM=6$. Notice also that $\triangle GBL$ is a $30-60-90$ right triangle, so $GL=6\sqrt{3}$. The area of $\triangle GBM$ is $\frac{1}{2}\cdot 6\cdot(6+6\sqrt{3})=\mathbf{18+18\sqrt{3}}$.

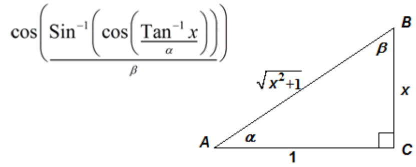
3. $\boxed{\boldsymbol{\frac{55}{73}}}$ Drop a perpendicular from Y to $\overline{RE}$, intersecting at Z. Then $YZ=RT/2=3$, $MY=YE=x$, and $ZE=8-x$. Applying the Pythagorean Theorem to $\triangle ZEY$, it follows that $3^2+(8-x)^2=x^2\rightarrow x=\frac{73}{16}$. Then it follows that $ZE=8-\frac{73}{16}=\frac{55}{16}$, so the answer is $\sin(\angle YEC)=\boldsymbol{\frac{55}{73}}$.

Team Round

1. $\boxed{\mathbf{36}}$ To be a multiple of 4, the number must end in 12, 16, 24, 52, 56, or 64. For each of these endings, there are $3\cdot 2=6$ ways to insert the initial two digits in each of the above integers to create the numbers in Patrick's list. Thus, the answer is $6\cdot 6=\mathbf{36}$.

2. $\boxed{\mathbf{31}}$ Note that $(ac+bc)-(ab+bc)=234-220=14=a(c-b)$. This implies that $a=1$ or $a=2$ or $a=7$ or $a=14$.
If $a=1$, then $c-b=14$, and also $c+b=168$, so $c=91$. Because there are no positive integers a and b such that $91(a+b)=234$, this results in a contradiction.
If $a=2$, then $c-b=7$, and also $c+b=84$, so $2c=91$. Because there is no positive integer c such that $2c=91$, this results in a contradiction.
If $a=14$, then $c-b=1$, and also $c+b=12$, so $2c=13$. Because there is no positive integer c such that $2c=13$, this results in a contradiction.
If $a=7$, then $c-b=2$, and also $c+b=24$, so $c=13$ and $b=11$. These values satisfy all three equations, so the answer is $7+11+13=\mathbf{31}$.

3. $\boxed{\sqrt{\mathbf{3}}}$ First, note that $\cos(\tan^{-1}x)=\dfrac{1}{\sqrt{1+x^2}}$ and that $\cos(\sin^{-1}u)=\sqrt{1-u^2}$. Consider the following diagram.



Therefore, the left-hand side of the given equation simplifies to $\sqrt{1 - \dfrac{1}{1+x^2}}$. Setting this equal to $\frac{x}{2}$ and solving yields $\dfrac{x^2}{1+x^2} = \frac{x^2}{4} \rightarrow 1 + x^2 = 4$, so the value of x is $\sqrt{\mathbf{3}}$.

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